Design of controllers in the complex domain

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***Abstract*— The paper discusses a class of symmetric systems that can be reduced to systems with half the dimension and order, but with complex coefficients. Symmetric systems include doubly-fed induction machines, self-excited induction generators, three-phase converters, and three-phase active filters. The paper reviews general properties of symmetric systems in the state-space and frequency domains. Although complex models have been used in the literature in the past, few papers have considered their practical use, in particular for the direct design of controllers in the complex domain. The paper illustrates such a design in the case of a doubly-fed induction machine.**

I. Introduction

For many years, researchers have recognized that certain systems encountered in energy applications could be transformed into equivalent systems of lower dimension but with complex parameters [19]. Such systems include several examples of electric machines and power electronic systems. More recently, researchers have discovered that the complex models could be used for estimation and control design with some advantages. In [17], for example, an extended complex Kalman filter (ECKF) was developed for sensorless induction motor control. In [1], a three-phase active power filter was constructed in the complex domain using H_∞ theory. In [18], an ECKF was developed with some advantages for the estimation of sinusoids. [15] similarly focused on frequency estimation, with applications in power systems. [6] obtained analytic conditions for spontaneous self-excitation of induction generators, and in [8], a controller was designed in the complex domain for a three-phase converter.

Dynamic systems with complex parameters have been considered in other fields, although to a limited extent as well. In [14], adaptation laws were developed for complex neural networks. In [2], a bandpass sigma delta modulator was developed in the complex domain, and performance improvements were shown over a modulator of the same order using real transfer functions. In [13], the use of complex polynomials in mobile communications was reviewed. The paper explains how the baseband representation of a received signal is nonsymmetric around the modulation frequency even if the transmitted sequence is real-valued. This fact explains why the transfer function of the channel has complex coefficients.

A limited number of control theoretical results have also been extended to systems with complex coefficients. In [4], [16], robust control theoretical results were extended to complex models, although without specific applications. Examples in power systems constitute possible applications to such results. The paper reviews general definitions and properties of systems that can be transformed into a complex representation. Then, the example of a doubly-fed induction machine is used to demonstrate how certain stability properties can be obtained more easily in the complex framework. The paper concludes with the design of a power control law for a doubly-fed induction machine using root locus plots of a system with complex parameters.

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II. Complex representation of symmetric systems

A. Symmetric systems

Definition - Symmetric system: a system that is described by the state-space model

$$\dot{x} = Ax + Bu, \quad y = Cx \tag{1}$$

is called *symmetric* if the state, input, and output vectors can be divided into two vectors of equal dimensions such that

$$x = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}, \quad u = \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix}, \quad y = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} \tag{2}$$

and the associated submatrices of A, B, and C have the structure

$$\begin{aligned} A &= \begin{pmatrix} A_{11} & -A_{21} \\ A_{21} & A_{11} \end{pmatrix}, \quad B = \begin{pmatrix} B_{11} & -B_{21} \\ B_{21} & B_{11} \end{pmatrix}, \\ C &= \begin{pmatrix} C_{11} & -C_{21} \\ C_{21} & C_{11} \end{pmatrix} \end{aligned} \tag{3}$$

Note that the submatrices of A must be square, but the submatrices of B and C may have different numbers of rows and columns. We adopt here the name of symmetric system following [12]. In [7], such systems were referred to as *isotropic*, or *rotational-invariant*. Indeed, consider the following property.

Fact - Rotational invariance: consider the transformation

$$U_n = \begin{pmatrix} \cos(\theta)\ I_n & -\sin(\theta)\ I_n \\ \sin(\theta)\ I_n & \cos(\theta)\ I_n \end{pmatrix} \tag{4}$$

where the angle θ is arbitrary and I_n denotes the identity matrix of dimension n. Define transformed variables

$$x' = U_n\ x,\ u' = U_m\ u,\ y' = U_p\ y \tag{5}$$

where n is the dimension of x_1 and x_2, m is the dimension of u_1 and u_2, and p is the dimension of y_1 and y_2. Then, the transformed variables are related through the *same state-space model* as the original system, *i.e.*,

$$\dot{x}' = Ax' + Bu', \quad y' = Cx' \tag{6}$$

Proof: the transformed matrices are given by

$$A' = U_n A U_n^{-1},\ \ B' = U_n B U_m^{-1},\ \ C' = U_p C U_n^{-1} \tag{7}$$

Given that $U_n^{-1} = U_n^T$, it is straightforward to verify that $A' = A$, $B' = B$, and $C' = C$.□

B. Complex representation of symmetric systems

For symmetric systems, a representation with half the number of states, inputs, and outputs can be obtained by defining complex vectors

$$x_c = x_1 + jx_2, \;\; u_c = u_1 + ju_2, \;\; y_c = y_1 + jy_2 \tag{8}$$

Indeed,

$$\dot{x}_c = A_c xc + B_c u_c, \quad y_c = C_c x_c \tag{9}$$

where

$$A_c = A_{11} + jA_{21}, \;\; B_c = B_{11} + jB_{21}, \;\; C_c = C_{11} + jC_{21} \tag{10}$$

We refer to this system as the *complex system*, while the original state-space model is called the *real system*. The following fact relates the poles of the real and complex systems.

Fact - Poles of the real and complex systems: any root of $\det(sI - A_c) = 0$ is a root of $\det(sI - A) = 0$. On the other hand, if s_0 is a root of $\det(sI - A) = 0$, then either s_0 or its complex conjugate s_0^* is a root of $\det(sI - A_c) = 0$. The fact is proved in [6]. It implies that, due to the special structure of the state-space model, the roots of $\det(sI - A) = 0$ must be either complex pairs or double real pairs. In other words, there cannot be single real roots. Further, each root of $\det(sI - A_c) = 0$ is a root of $\det(sI - A) = 0$, and each root of $\det(sI - A) = 0$ is represented in $\det(sI - A_c) = 0$, either as itself or as its complex conjugate. Thus, knowledge of the eigenvalues of A_c is equivalent to knowledge of the eigenvalues of A: all the poles of the real system can be obtained from the poles of the complex system and *vice-versa*.

C. Transfer function of the complex representation

As found in the following fact, the partitioning of the state-space model implies a similar partitioning of the transfer function matrix, as well as a simple relationship between the transfer function matrices of the real and complex systems.

Fact - Transfer function matrices of the real and complex systems: the transfer function matrix from u to y can be partitioned similarly to the state-space model, with

$$H(s) = C(sI - A)^{-1}B = \begin{pmatrix} H_{11}(s) & -H_{21}(s) \\ H_{21}(s) & H_{11}(s) \end{pmatrix} \tag{11}$$

The transfer function matrix from u_c to y_c of the complex system is

$$H_c(s) = C_c(sI - A_c)^{-1}B_c \tag{12}$$

where A_c, B_c, and C_c are given in (10). $H_c(s)$ is also equal to

$$H_c(s) = H_{11}(s) + jH_{21}(s) \tag{13}$$

where $H_{11}(s)$ and $H_{21}(s)$ are the sub-matrices of (11).
The fact is proved in [8]. Note that the denominators of $H_{11}(s)$ and $H_{22}(s)$ originate from the same polynomial $\det(sI - A)$ which is of order $2n$. However, the denominator of $H_c(s)$ also originates from $\det(sI - A_c)$ and can be at most of order n. Therefore, n poles must be cancelled either in the computation of $H_{11}(s)$ and $H_{12}(s)$, or in the computation of $H_c(s) = H_{11}(s) + jH_{21}(s)$.

D. Systems with 2 inputs and 2 outputs

Systems with 2 inputs and 2 outputs consititute a special case where the complex system becomes a single-input single-output system. In particular, one can write the transfer function $H_c(s)$ as $H_c(s) = N_h(s)/D_h(s)$, where N_h and D_h are the numerator and denominator polynomials of $H_c(s)$, which have complex coefficients. For control, it is not unreasonable to restrict oneself to control systems that are symmetric, so that they can also be represented in the complex formulation

$$u_c(s) = G_c(s)(r_c(s) - y_c(s)) \tag{14}$$

where r_c is the complex reference input. The closed-loop poles are then given by the roots of the polynomial with complex coefficients

$$D_g(s)D_h(s) + N_g(s)N_h(s) = 0 \tag{15}$$

where $G_c(s) = N_g(s)/D_g(s)$.

E. Root-locus design

It is possible to design controllers for 2×2 systems using root locus plots of the complex system. Specifically, define

$$D_{ol}(s) = D_g(s)D_h(s) = (s - p_1)(s - p_2)\ldots(s - p_n) \tag{16}$$

where $p_1, p_2, \ldots, p_n$ are the poles of the open-loop system, and

$$N_g(s)N_h(s) = k_g k_h N_{ol}(s) \tag{17}$$

where k_g is the complex gain of the controller, k_h is the complex gain of the plant, and

$$N_{ol}(s) = (s - z_1)(s - z_2)\ldots(s - z_m) \tag{18}$$

where $z_1, z_2, \ldots, z_m$ are the zeros of the open-loop system. For a root locus design, one inserts an additional, real gain $k > 0$ so that the closed-loop poles of the complex system are determined by the roots of

$$D_{cl}(s) = D_{ol}(s) + kk_g k_h N_{ol}(s). \tag{19}$$

By definition, the *complex root locus* is the locus of the roots of $D_{cl}(s) = 0$ as k varies from 0 to infinity. Because the coefficients of the polynomials are not necessarily real, the poles do not have to appear as complex pairs. However, the gain k appears linearly in (19). In contrast, the complex roots of the real system appear in complex pairs, but their patterns does not satisfy typical root locus rules because the characteristic polynomial of the real system is typically *nonlinear* in the parameter k. The real system involves a *multivariable* feedback law.

In [19], poles and zeros of complex transfer functions were computed, and root loci were plotted. However, rules were not derived for the complex root locus. Such rules were derived in [8], and it turns out that most of the rules of the conventional root locus also apply to the root locus for systmes with complex coefficients, either directly or with minor adjustments. Yet, some peculiar differences also exist:

- the root locus is not necessarily symmetric with respect to the real axis;
- the angles of the asymptotes do not satisfy the symmetry conditions of the real root locus;
- no portion of the real axis typically belongs to the root locus;
- break-away points are less common than in the real root locus (because of the previous statement);
- with a complex gain in the controller, an additional degree of freedom can be used in control design.

Complex root locus plots are shown in a section to follow. The root locus of the real system can then be obtained from the complex root locus by combining the complex root locus with its mirror image with respect to the real axis (doubling the multiplicity of poles on the real axis).

III. Example a doubly-fed induction machine (DFIM)

A. Real and complex models of a DFIM

Doubly-fed induction machines are often used as generators in wind farms. For convenience, a three-phase to two-phase transformation is used to reduce the model to a two-phase machine. For example, in a power-preserving transformation, the three-phase currents $i_{S\bar{A}}$, $i_{S\bar{B}}$, and $i_{S\bar{C}}$ are transformed into the two-phase variables i_{SA}, i_{SB} plus a homopolar current i_{Sh} (that is typically zero) using

$$\begin{pmatrix} i_{SA} \\ i_{SB} \\ i_{Sh} \end{pmatrix} = \sqrt{\frac{2}{3}} \begin{pmatrix} 1 & -1/2 & -1/2 \\ 0 & \sqrt{3}/2 & -\sqrt{3}/2 \\ 1/\sqrt{2} & 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} i_{S\bar{A}} \\ i_{S\bar{B}} \\ i_{S\bar{C}} \end{pmatrix} \tag{20}$$

The rotor currents, and the rotor and stator voltages are transformed similarly.

The electrical variables describing the resulting, equivalent two-phase machine, satisfy the equations

$$\begin{pmatrix} L_S & 0 & M & 0 \\ 0 & L_S & 0 & M \\ M & 0 & L_R & 0 \\ 0 & M & 0 & L_R \end{pmatrix} \frac{d}{dt} \begin{pmatrix} i_{SF} \\ i_{SG} \\ i_{RF} \\ i_{RG} \end{pmatrix}$$
$$= \begin{pmatrix} v_{SF} - R_S\, i_{SF} + \omega_F (L_S i_{SG} + M i_{RG}) \\ v_{SG} - R_S\, i_{SG} - \omega_F (L_S i_{SF} + M i_{RF}) \\ v_{RF} - R_R\, i_{RF} - (n_p\omega - \omega_F)(L_R i_{RG} + M i_{SG}) \\ v_{RG} - R_R\, i_{RG} + (n_p\omega - \omega_F)(L_R i_{RF} + M i_{SF}) \end{pmatrix} \tag{21}$$

The parameters of the model are L_S, the self-inductance of a stator winding (H), L_R, the self-inductance of a rotor winding (H), M, the mutual inductance between a stator winding and a rotor winding when they are aligned (H), R_S, the resistance of a stator winding (Ω), R_R, the resistance of a rotor winding (Ω), and n_P, the number of pole pairs. The variable ω is the mechanical speed of the rotor (rad/s) and is also treated as a parameter in the following derivations. The variables v_{SF}, v_{SG} and v_{RF}, v_{RG} are the stator and rotor voltages (V), respectively. Similarly, i_{SF}, i_{SG} and i_{RF}, i_{RG} are the stator and rotor currents (A). The variables are expressed in an arbitrary frame of reference with axes F and G, where the angle of the F axis with respect to the A axis of the stator is θ_F. The speed of rotation of the F axis is $\omega_F = d\theta_F/dt$. In the FG frame of reference, the stator currents are given by

$$\begin{pmatrix} i_{SF} \\ i_{SG} \end{pmatrix} = \begin{pmatrix} \cos(\theta_F) & \sin(\theta_F) \\ -\sin(\theta_F) & \cos(\theta_F) \end{pmatrix} \begin{pmatrix} i_{SA} \\ i_{SB} \end{pmatrix} \tag{22}$$

where i_{SA}, i_{SB} are the currents in the two-phase stator windings. The rotor currents i_{RF}, i_{RG} are given by

$$\begin{pmatrix} i_{RF} \\ i_{RG} \end{pmatrix} = \begin{pmatrix} \cos(\delta) & \sin(\delta) \\ -\sin(\delta) & \cos(\delta) \end{pmatrix} \begin{pmatrix} i_{RX} \\ i_{RY} \end{pmatrix} \tag{23}$$

where i_{RX}, i_{RY} are the two-phase rotor currents, $\delta = \theta_F - n_P\theta$, and θ is the angle of the rotor (with $d\theta/dt = \omega$). The stator voltages v_{SF}, v_{SG}, and the rotor voltages v_{RF}, v_{RG} are defined similarly from the physical voltages.

Several models can be obtained as special cases of (21). In particular,

- for $\theta_F = 0$, the model becomes the model in the *stator frame of reference* (replacing F, G by A, B), also called the *$\alpha\beta$ model*.
- for $\theta_F = n_p\theta$, the model becomes the model in the *rotor frame of reference* (rarely used, but sometimes useful).
- for θ_F such that $\psi_{RF} = L_R i_{RF} + M i_{SF} = 0$, the model becomes the *flux-oriented DQ* model.
- for θ_F equal to the angle of the vector (v_{SA}, v_{SB}), the model becomes the model in a stator voltage reference frame, also called *grid-oriented* or *synchronous DQ* model. This model is most useful for a grid-connected machine.

The equations for the doubly-fed induction machine satisfy the symmetry conditions, so that the model can be transformed into the complex representation by defining

$$\begin{aligned} v_S &= v_{SF} + jv_{SG}, \quad i_S = i_{SF} + ji_{SG}, \\ v_R &= v_{RF} + jv_{RG}, \quad i_R = i_{RF} + ji_{RG} \end{aligned} \tag{24}$$

With these definitions, the model (21) becomes

$$\begin{pmatrix} L_S & M \\ M & L_R \end{pmatrix} \frac{d}{dt} \begin{pmatrix} i_S \\ i_R \end{pmatrix}$$
$$= \begin{pmatrix} v_S - R_S i_S - j\omega_F (L_S i_S + M i_R) \\ v_R - R_R i_R + j(n_P\omega - \omega_F)(L_R i_R + M i_S) \end{pmatrix} \tag{25}$$

These equations are sometimes used to compute the steady-state responses of the induction machine. For stator voltages with constant frequency ω_F, steady-state currents can be computed by replacing d/dt by $j\omega_F$. Then, v_S, v_R, i_S, i_R can be interpreted as phasors. However, the use of (25) is not restricted to steady-state analysis. Indeed, the real and complex models are equivalent, and the complex model (25) is simply a more compact representation.

B. Analysis of open-loop stability

Although complex models are frequently found in the literature on electric machines and power systems, they are rarely used for anything more than to simplify the notation.

However, much can be done using the complex models. For example, suppose that we were to ask a simple theoretical question: is the doubly-fed induction machine model stable for all possible motor parameters and for any fixed speed? With the real model and assuming $\omega_F = 0$ (for simplicity), stability is determined by the values of s such that

$$\det \begin{pmatrix} sL_S + R_S & 0 & sM & 0 \\ 0 & sL_S + R_S & 0 & sM \\ sM & n_p\omega M & sL_R + R_R & n_p\omega L_R \\ -n_p\omega M & sM & -n_p\omega L_R & sL_R + R_R \end{pmatrix} = 0 \quad (26)$$

The stability question can be answered by applying the Routh-Hurwitz test to the polynomial of degree 4

$$c_0 s^4 + c_1 s^3 + c_2 s^2 + c_3 s + c_4 = 0 \quad (27)$$

where

$$\begin{aligned} c_0 &= \mu^2, \;\; c_1 = 2\mu(L_S R_R + L_R R_S) \\ c_2 &= (L_S R_R + L_R R_S)^2 + 2\mu R_S R_R + (n_p\omega)^2\mu^2 \\ c_3 &= 2R_S(R_R(L_S R_R + L_R R_S) + (n_p\omega)^2\mu L_R) \\ c_4 &= R_S^2(R_R^2 + (n_p\omega L_R)^2) \\ \mu &= L_S L_R - M^2 \end{aligned} \quad (28)$$

The Routh -Hurwitz test requires that $c_0 > 0$, $c_1 > 0$, $c_4 > 0$, $c_1c_2 - c_0c_3 > 0$, and $c_1c_2c_3 - c_0c_3^2 - c_1^2c_4 > 0$. The first three inequalities are trivially satisfied, while the last two are quite tedious to compute manually. Using the *Matlab Symbolic Toolbox* and the function *simplify* (as well as the function *simple* for the first term) gives

$$\begin{gathered} c_1c_2 - c_0c_3 = 2\mu(L_R^3R_S^3 + 3L_R^2L_SR_RR_S^2 \\ +3L_RL_S^2R_R^2R_S + \mu L_RR_RR_S^2 + L_S^3R_R^3 \\ +\mu^2(n_p\omega)^2L_SR_R + \mu L_SR_R^2R_S) \\ c_1c_2c_3 - c_0c_3^2 - c_1^2c_4 = 4\mu R_RR_S \\ (L_R^2R_S^2 + 2L_RL_SR_RR_S + L_S^2R_R^2 + \mu^2(n_p\omega)^2) \\ (L_R^2R_S^2 + \mu L_RL_S(n_p\omega)^2 + 2L_RL_SR_RR_S + L_S^2R_R^2) \end{gathered} \quad (29)$$

Both terms are positive, so that stability can be concluded.

In contrast, stability is determined in the complex domain by the roots of

$$\det \begin{pmatrix} sL_S + R_S & sM \\ (s - jn_P\omega)M & (s - jn_P\omega)L_R + R_R \end{pmatrix} = a_0s^2 + (a_1 + jb_1)s + (a_2 + jb_2) = 0 \quad (30)$$

where

$$\begin{aligned} a_0 &= \mu, \; a_1 = L_SR_R + L_RR_S, \; b_1 = -n_P\omega\mu, \\ a_2 &= R_SR_R, \; b_2 = -n_P\omega R_SL_R \end{aligned} \quad (31)$$

Stability can be determined using a Hurwitz test for a polynomial of degree 2 with complex coefficients, instead of a polynomial of degree 4 with real coefficients. Although most control engineers are only familiar with the Routh-Hurwitz test for real coefficients, the test for complex coefficients is available from the literature [10]. Stability is guaranteed for the complex polynomial if and only if $a_0 > 0$, $a_1 > 0$, and

$$\Delta_2 = \begin{vmatrix} a_1 & 0 & -b_2 \\ a_0 & a_2 & -b_1 \\ 0 & b_2 & a_1 \end{vmatrix} = R_SR_R\left((L_SR_R + L_RR_S)^2 + \mu L_SL_R(n_P\omega)^2\right) > 0 \quad (32)$$

Since all three quantities are positive for physical parameter values, one can again conclude that the model is stable for all speeds. However, the result is obtained this way after only a few simple computations.

While the computations required were tractable in the real domain for the 4^{th} order model using a symbolic computation software, cases involving 6^{th} order models have been found too complicated to handle. In [3], the authors could not find conditions on some proportional and integral gains that would guarantee the stability of the proposed control law. In contrast, analysis of the associated complex third-order polynomial [5] quickly revealed the necessary and sufficient condition for the stability of the closed-loop system, specifically

$$k_I < \frac{k_P^2R_SML_R}{\mu(\mu\omega_S + k_PM)} \quad (33)$$

where k_P and k_I were the proportional and integral gains of the controller. Only through the complex analysis could this remarkably simple condition be discovered.

Conditions were also found for spontaneous self-excitation of squirrel-cage induction generators in [6]. Self-excited induction generators (SEIG) are useful for the production of power from renewable sources in remote areas and in developing countries. The results of [6] were not previously known because the real models of dimension 6 prevented any type of analysis of stability.

C. Control design in the complex domain

1) Control law #1: The complex analysis can be used to design control laws in a manner similar to the conventional root locus design, but with some interesting adjustments and possibilities. For example, consider the control law of [3], which consists in setting the complex rotor voltages of (25) to

$$\begin{aligned} v_R &= \alpha R_Ri_R - j(n_P\omega - \omega_F)(L_Ri_R + Mi_S) \\ &+jk_P(i_{S,REF} - i_S) + jk_I\int_0^t((i_{S,REF} - i_S)d\tau \end{aligned} \quad (34)$$

with $\alpha = 1$. The complex variable $i_{S,REF}$ is the reference value for the stator currents. Assuming a direct connection to the grid, the regulation of the current i_S is equivalent to the control of the active and reactive powers produced by the doubly-fed induction machine. The frequency ω_F is set equal to the grid frequency, so that all variables are constant in steady-state.

The overall system is described in the Laplace domain by

$$\begin{pmatrix} (s+j\omega_F)\,L_S+R_S & (s+j\omega_F)\,M & 0 \\ sM+jk_P & sL_R & -jk_I \\ 1 & 0 & s \end{pmatrix}\begin{pmatrix} i_S \\ i_R \\ x_I \end{pmatrix} = \begin{pmatrix} v_S \\ jk_P i_{S,REF} \\ i_{S,REF} \end{pmatrix} \quad (35)$$

where, in the time domain

$$x_I = \int_0^t ((i_{S,REF} - i_S)d\tau \quad (36)$$

The poles of the system are given by the roots of a polynomial of the form (19), where

$$\begin{aligned} D_{ol}(s) &= s^2\left(s+\frac{R_S L_R}{\mu}+j\omega_F\right) \\ N_{ol}(s) &= (s+j\omega_F)\left(s+\frac{k_I}{k_P}\right) \\ k_h &= \frac{M}{\mu},\;\; k=k_P,\;\; k_g=-j \end{aligned} \quad (37)$$

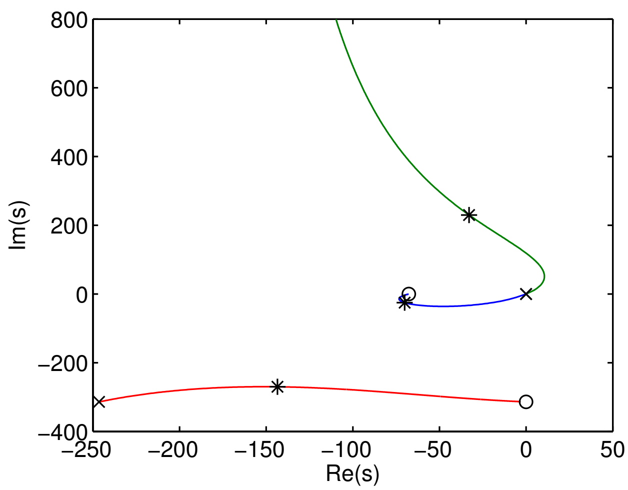

Fig. 1. Root locus of control law #1

The root locus of the system is shown in Fig. 1 for the values of [9]: $R_S = 4.92\Omega$, $R_R = 4.42\Omega$, $L_S = 0.725H$, $L_R = 0.715H$, $M = 0.71H$, and $\omega_F = 314$ rad/s. Note that there are two open-loop poles at $s = 0$ and one at $s = -246 - 314j$. The two zeros are at $s = -314j$ and $s = -k_I/k_P = -67.7$. This last value was found to be a good setting for the controller zero. The root locus plot shows that the poles of the closed-loop system are stable for a sufficiently large gain k, confirming the prediction of (33). One of the poles moves towards a 90° asymptote. For a system with a number of poles minus number of zeros equal to 1, the asymptote is parallel to the complex number $-k_h k_g$, *i.e.,* parallel to $s = j$. The $*$'s on the plot show pole locations for a satsifactory control system design, corresponding to $k_P = 5$.

2) Control law #2: While control law #1 produces a stable closed-loop system, it exhibits two undesirable features:

- the system is only conditionally stable (it becomes unstable if the gain is reduced);

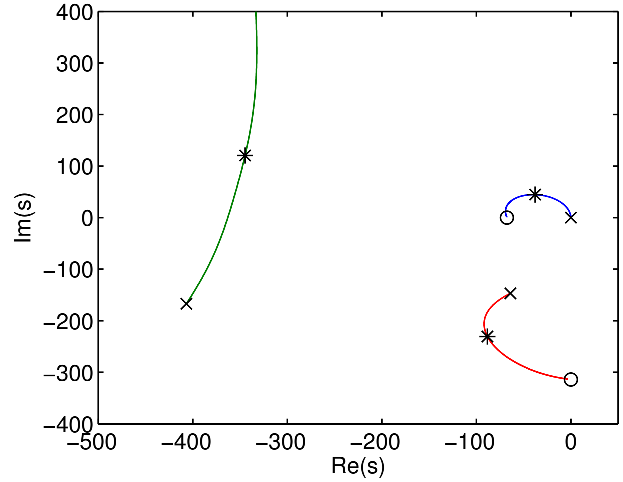

Fig. 2. Root locus for control law #2

- two poles are poorly damped, no matter what gain is chosen.

The conditional stability of the system is due to the double pole at $s = 0$. One of these poles comes from the integrator in the control law, while the other originates from the cancellation of the rotor resistance in (34). This cancellation removes the natural damping produced by the rotor resistance, and the stability of the induction machine model deduced from the earlier analysis. This observation suggests letting $\alpha = 0$ in the control law (34) so that the effective rotor resistance is not reduced to zero. This second control law gives a new denominator polynomial

$$\begin{aligned} D_{ol}(s) &= s^3+\left(\frac{R_S L_R + R_R L_S}{\mu}+j\omega_F\right)s^2 \\ &\quad +(R_S+j\omega_F L_S)\,\frac{R_R}{\mu}s \end{aligned} \quad (38)$$

while $N_{ol}(s)$ remains the same. With this modification, the open-loop system has only one pole at $s = 0$, while the other two poles are stable.

The root locus for control law #2 is shown in Fig. 2. The $*$'s correspond to a possible design with $k_P = 5$. Note that the system is now stable for all gains. On the other hand, one of the poles remains poorly damped, no matter what gain is chosen (red line of the root locus).

The complex nature of k_g gives an additional degree of freedom that can be used in the design, but it turns out not to be sufficient to provide adequate damping the poles. Fig. 3 shows the root locus for $k_g = 1$ instead of $k_g = -j$. As expected, the asymptote becomes parallel to the real axis, but the poorly damped pole becomes even less damped.

3) Control law #3: To improve the damping of the closed-loop design, one must increase the damping of the open-loop pole. One would expect that this could be achieved by increasing the effective resistance of the rotor winding, *i.e.*, but setting $\alpha < 0$ in the control law (34). Surprisingly, the reverse turns out to be true: with $\alpha < 0$, the complex pole moves even closer to the imaginary axis for $\alpha < 0$. Note that $\alpha = 1$ corresponds to control law #1, which cancels the rotor

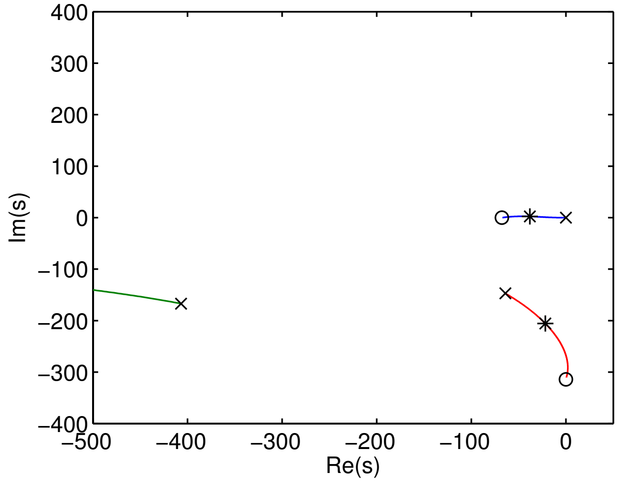

Fig. 3. Root locus for control law #2 with $k_g = 1$

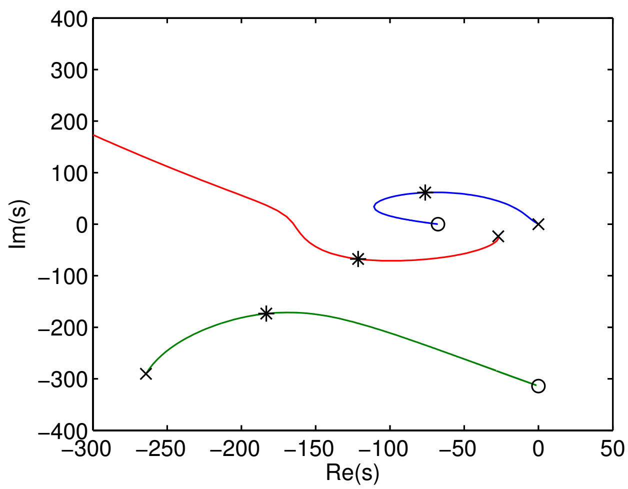

Fig. 4. Root locus for control law #3

resistance and is undesirable because the second pole moves to the origin. It was found that good results were obtained with $\alpha = 0.8$, which corresponds to reducing the effective rotor resistance to 20% of its actual value. Fig. 4 shows the root locus for $\alpha = 0.8$, and $k_g = 1 - 1.5j$. The $*$'s correspond to a possible design with $k_P = 1.8$. The poles are then given by $-76 + 61j$, $-121 - 68j$, and $-183 - 174j$, which are all adequately damped. Note that the real root locus is the union of the complex root locus and its mirror image. The real root locus has branches that cross each other without being breakaway points, which is not possible in the conventional root locus. This is because the system is a multivariable system, with a characteristic polynomial that is not linear in k.

IV. Conclusions

The paper reviewed properties of symmetric systems and tools that can be used to analyze them through equivalent complex systems with half the dimension or order. The control of a doubly-fed induction machine was used as an example. It was shown that certain stability guarantees could be more easily obtained in the complex domain than in the real domain. The design of a controller using root locus plots in the complex domain was also described in detail. No equivalent design technique exists in the real domain, and it was shown that a satisfactory placement of closed-loop poles could be obtained after a few iterations using an understanding of root locus properties.

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