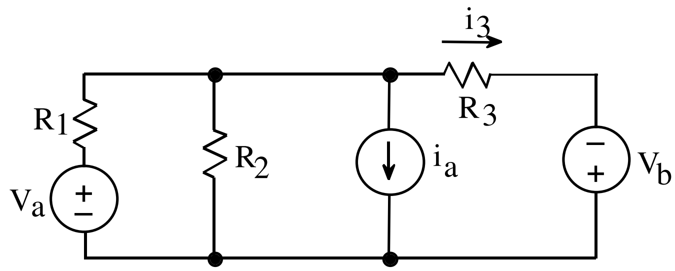
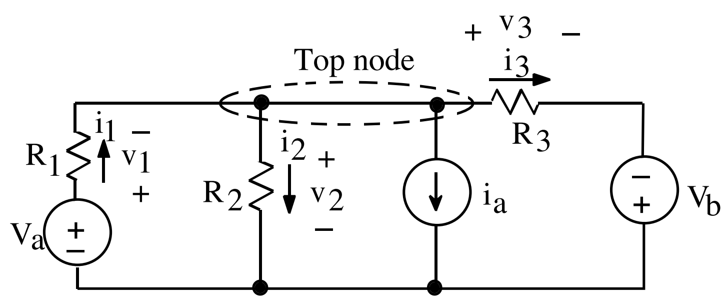
2. (30 points)

Derive an expression for i_3. The expression must not contain more than the circuit parameters V_a, V_b, i_a, R_1, R_2, and R_3.



ans: $$i_3 = \frac{V_a R_2 + V_b(R_1 + R_2) - i_a R_1 R_2}{R_1 R_2 + R_3(R_1 + R_2)}$$

sol'n: Using passive sign convention, label voltage drop and current measurement polarities.



Use Kirchhoff's laws:

sum v drops around loop = 0
sum i out of node = 0

v drops for loop on left, using Ohm's law for v_1 and v_2:

$$V_a - i_1 R_1 - i_2 R_2 = 0 \text{ V}$$

Middle loop would include current source, so use slightly larger loop with R_2 on left and V_b on right:

$$i_2 R_2 - i_3 R_3 + V_b = 0 \text{ V}$$

Now sum currents out of top node (that consists of the two top nodes connected by a wire). Note: We are always allowed to combine nodes connected by wires.

$$-i_1 + i_2 + i_a + i_3 = 0 \text{ A}$$

We now have three equations in three unknowns. We solve for i_3. Use the second equation to eliminate i_2:

$$i_2 = \frac{i_3 R_3 - V_b}{R_2}$$

Use the first equation to eliminate i_1:

$$i_1 = \frac{V_a - i_2 R_2}{R_1} = \frac{1}{R_1}\left[V_a - \left(\frac{i_3 R_3 - V_b}{\cancel{R_2}}\right)\cancel{R_2}\right] = \frac{1}{R_1}\left(V_a + V_b - i_3 R_3\right)$$

Substitute for i_1 and i_2 in the third equation:

$$-\frac{1}{R_1}\left(V_a + V_b - i_3 R_3\right) + \frac{i_3 R_3 - V_b}{R_2} + i_a + i_3 = 0 \text{ A}$$

Solve for i_3:

$$-\frac{1}{R_1}\left(V_a + V_b\right) - \frac{V_b}{R_2} + i_a + \frac{1}{R_1} i_3 R_3 + \frac{i_3 R_3}{R_2} + i_3 = 0 \text{ A}$$

$$i_3\left(\frac{R_3}{R_1} + \frac{R_3}{R_2} + 1\right) = \frac{1}{R_1}\left(V_a + V_b\right) + \frac{V_b}{R_2} - i_a$$

Multiply both sides by R_1R_2 to clear fractions:

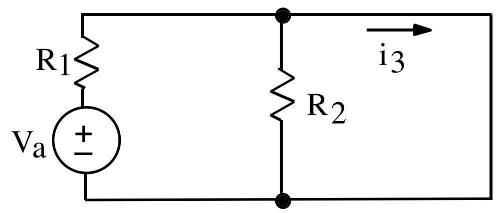
$$i_3\left(R_3R_2 + R_3R_1 + R_1R_2\right) = R_2\left(V_a + V_b\right) + R_1V_b - i_aR_1R_2$$

or

$$i_3 = \frac{V_aR_2 + V_b\left(R_1 + R_2\right) - i_aR_1R_2}{R_1R_2 + R_3\left(R_1 + R_2\right)}$$

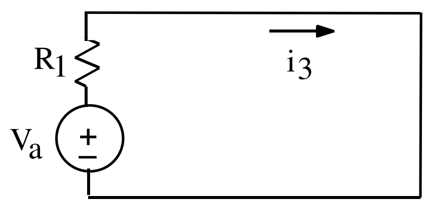
Now for consistency checks to verify our answer. (Optional)

1) Consider $i_a = 0$, $V_b = 0$, and $R_3 = 0$:



$V_b = 0$ (short)

Since R_2 is bypassed by a short, no current flows in R_2. Therefore, we can remove R_2 without changing i_3:



$i_3 = \frac{V_a}{R_1}$ by Ohm's law

Our formula gives $i_3 = \frac{V_a}{R_1}$. ✓

2) Consider $i_a = 0$ (open circuit) and $R_2 = \infty$ (open circuit):

Removing R_2 and i_a leaves total voltage $V_a + V_b$ across $R_1 + R_3$ in outside loop.

Therefore, we have

$$i_3 = \frac{(V_a + V_b)}{(R_1 + R_3)}$$

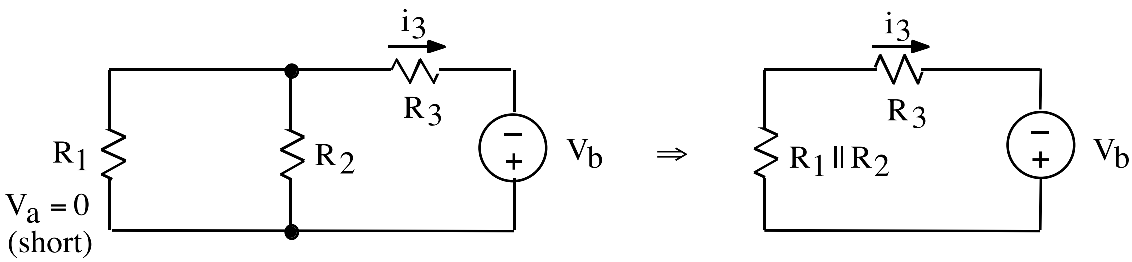
For our formula, we use the following identities:

$$\lim_{R_2 \to \infty} \frac{R_2}{R_1 R_2 + R_3(R_1 + R_2)} = \frac{1}{R_1 + R_3} \qquad \lim_{R_2 \to \infty} \frac{R_1 + R_2}{R_1 R_2 + R_3(R_1 + R_2)} = \frac{1}{R_1 + R_3}$$

Making these substitutions in our formula gives

$$i_3 = \frac{(V_a + V_b)}{(R_1 + R_3)}$$ ✓

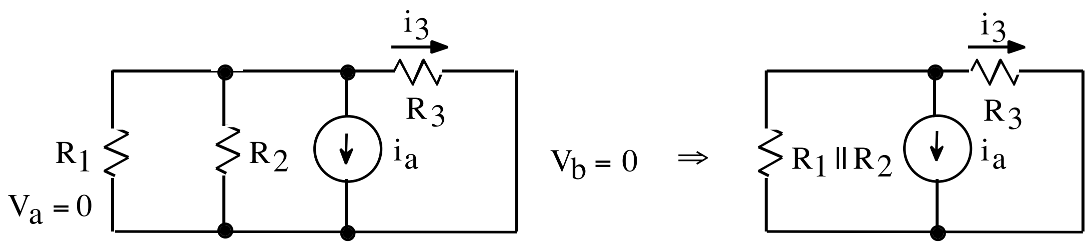
(3) Consider $V_a = 0$, $i_a = 0$:



$$i_3 = \frac{V_b}{R_1 \| R_2 + R_3}$$

Our formula gives $i_3 = \dfrac{V_b(R_1 + R_2)}{R_1R_2 + R_3(R_1 + R_2)}$ or $i_3 = \dfrac{V_b}{R_1 \| R_2 + R_3}$. ✓

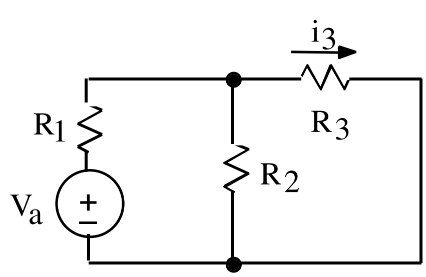
(4) Consider $V_a = 0$, $V_b = 0$:



By current divider formula, we have $i_3 = -\dfrac{i_a \, R_1 \| R_2}{R_1 \| R_2 + R_3}$.

Our formula gives $i_3 = -\dfrac{i_a R_1 R_2}{R_1R_2 + R_3(R_1 + R_2)}$ or $i_3 = -\dfrac{i_a \, R_1 \| R_2}{R_1 \| R_2 + R_3}$ ✓

(5) Consider $i_a = 0$, $V_b = 0$.



v-divider gives v across $R_2 \| R_3$

$$v_3 = V_a \frac{R_2 \| R_3}{R_2 \| R_3 + R_1}$$

Now

$$i_3 = \frac{V_3}{R_3} = \frac{V_a \, R_2 \| R_3}{R_3(R_1 + R_2 \| R_3)} = \frac{V_a \dfrac{R_2R_3}{R_2 + R_3}}{R_3\left(R_1 \dfrac{R_2 + R_3}{R_2 + R_3} + \dfrac{R_2R_3}{R_2 + R_3}\right)} = \frac{V_a \, R_2}{R_1(R_2 + R_3) + R_2R_3}$$

Our formula gives $i_3 = \dfrac{V_a R_2}{R_1R_2 + R_3(R_1 + R_2)}$. ✓