Introduction to Bode Plot

- 2 plots – both have logarithm of frequency on x-axis
  - y-axis magnitude of transfer function, H(s), in dB
  - y-axis phase angle

The plot can be used to interpret how the input affects the output in both magnitude and phase over frequency.

Where do the Bode diagram lines come from?
1) Determine the Transfer Function of the system:

\[ H(s) = \frac{K(s + z_1)}{s(s + p_1)} \]

2) Rewrite it by factoring both the numerator and denominator into the standard form

\[ H(s) = \frac{Kz_1 \left( \frac{s}{z_1} + 1 \right)}{sp_1 \left( \frac{s}{p_1} + 1 \right)} \]

where the \( z \) s are called zeros and the \( p \) s are called poles.

3) Replace \( s \) with \( j\omega \). Then find the **Magnitude** of the Transfer Function.

\[ H(j\omega) = \frac{Kz_1 \left( \frac{j\omega}{z_1} + 1 \right)}{jwp_1 \left( \frac{j\omega}{p_1} + 1 \right)} \]

If we take the \( \log_{10} \) of this magnitude and multiply it by 20 it takes on the form of

\[ 20 \log_{10} (H(j\omega)) = 20 \log_{10} \left( \frac{Kz_1 \left( \frac{j\omega}{z_1} + 1 \right)}{jwp_1 \left( \frac{j\omega}{p_1} + 1 \right)} \right) = 20 \log_{10} |K| + 20 \log_{10} |z_1| + 20 \log_{10} \left| \frac{j\omega}{z_1} + 1 \right| - 20 \log_{10} |p_1| - 20 \log_{10} |j\omega| - 20 \log_{10} \left| \frac{j\omega}{z_1} + 1 \right| \]

Each of these individual terms is very easy to show on a logarithmic plot. The entire Bode log magnitude plot is the result of the superposition of all the straight line terms. This means with a little practice, we can quickly see the effect of each term and quickly find the overall effect. To do this we have to understand the effect of the different kinds of terms.

These include: 1) **Constant terms** \( K \)

2) **Poles and Zeros at the origin** \( |j\omega| \)

3) **Poles and Zeros not at the origin** \( \left| 1 + \frac{j\omega}{p_1} \right| \) or \( \left| 1 + \frac{j\omega}{z_1} \right| \)

4) **Complex Poles and Zeros** (addressed later)
**Effect of Constant Terms:**
Constant terms such as $K$ contribute a straight horizontal line of magnitude $20 \log_{10}(K)$

\[
20 \log_{10}(H) = 20 \log_{10}(K)
\]

**Effect of Individual Zeros and Poles at the origin:**
A zero at the origin occurs when there is an $s$ or $j\omega$ multiplying the numerator. Each occurrence of this causes a positively sloped line passing through $\omega = 1$ with a rise of $20$ dB over a decade.

\[
H = |j\omega|
\]

A pole at the origin occurs when there are $s$ or $j\omega$ multiplying the denominator. Each occurrence of this causes a negatively sloped line passing through $\omega = 1$ with a drop of $20$ dB over a decade.

**Effect of Individual Zeros and Poles Not at the Origin**
Zeros and Poles not at the origin are indicated by the $(1+j\omega / z_i)$ and $(1+j\omega / p_i)$. The values $z_i$ and $p_i$ in each of these expression is called a critical frequency (or break frequency). Below their critical frequency these terms do not contribute to the log magnitude of the overall plot. Above the critical frequency, they represent a ramp function of $20$ dB per decade. Zeros give a positive slope. Poles produce a negative slope.

\[
H = \frac{1 + \frac{j\omega}{z_i}}{1 + \frac{j\omega}{p_i}}
\]
To complete the log magnitude vs. frequency plot of a Bode diagram, we superposition all the lines of the different terms on the same plot.

**Example 1:**
For the transfer function given, sketch the Bode log magnitude diagram which shows how the log magnitude of the system is affected by changing input frequency. \((TF=\text{transfer function})\)

\[ TF = \frac{1}{2s + 100} \]

Step 1: Repose the equation in Bode plot form:

\[ TF = \frac{\left( \frac{1}{100} \right)}{\frac{s}{50} + 1} \quad \text{recognized as} \quad TF = \frac{K}{\frac{1}{s + p_1}} \]

with \( K = 0.01 \) and \( p_1 = 50 \)

For the constant, \( K \):
\[ 20 \log_{10}(0.01) = -40 \]

For the pole, with critical frequency, \( p_1 \):

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**Example 2:**
Your turn. Find the Bode log magnitude plot for the transfer function,

\[ TF = \frac{5 \times 10^4 s}{s^2 + 505s + 2500} \]

Start by simplifying the transfer function form:
Example 2 Solution:
Your turn. Find the Bode log magnitude plot for the transfer function,

\[ TF = \frac{5 \times 10^4 s}{s^2 + 505s + 2500} \]

Simplify transfer function form:

\[ TF = \frac{5 \times 10^4 \frac{s}{(s + 5)(s + 500)}}{\frac{s}{5 + 1} + \frac{s}{500} + 1} = \frac{20s}{5 + 1} \]

Recognize: \( K = 20 \Rightarrow 20 \log_{10}(20) = 26.02 \)

1 zero at the origin

2 poles: at \( p_1 = 5 \) and \( p_2 = 500 \)

Technique to get started:
1) Draw the line of each individual term on the graph
2) Follow the combined pole-zero at the origin line back to the left side of the graph.
3) Add the constant offset, \( 20 \log_{10}(K) \), to the value where the pole/zero at the origin line intersects the left side of the graph.
4) Apply the effect of the poles/zeros not at the origin, working from left (low values) to right (higher values) of the poles/zeros.
Example 3: One more time. This one is harder. Find the Bode log magnitude plot for the transfer function,

\[ TF = \frac{200(s + 20)}{s(2s + 1)(s + 40)} \]

Simplify transfer function form:
Technique to get started:
1) Draw the line of each individual term on the graph
2) Follow the combined **pole-zero at the origin** line back to the left side of the graph.
3) Add the constant offset, $20 \log_{10}(K)$, to the value where the **pole/zero at the origin** line intersects the left side of the graph.
4) Apply the effect of the **poles/zeros not at the origin**, working from left (low values) to right (higher values) of the poles/zeros.

**Example 3 Solution:** Find the Bode log magnitude plot for the transfer function,

$$TF = \frac{200(s + 20)}{s(2s + 1)(s + 40)}$$

Simplify transfer function form:

$$TF = \frac{200(s + 20)}{s(2s + 1)(s + 40)} = \frac{200 \times 20}{40} \frac{(s + 1)}{(s + 10)} = \frac{100 \times (s + 1)}{s(0.5s + 1)(s + 40)}$$

Recognize: $K = 100 \rightarrow 20 \log_{10}(100) = 40$

- 1 pole at the origin
- 1 zero at $z_1 = 20$
- 2 poles: at $p_1 = 0.5$ and $p_2 = 40$
Technique to get started:
1) Draw the line of each individual term on the graph
2) Follow the combined pole-zero at the origin line back to the left side of the graph.
3) Add the constant offset, \(20 \log_{10}(K)\), to the value where the pole/zero at the origin line intersects the left side of the graph.
4) Apply the effect of the poles/zeros not at the origin, working from left (low values) to right (higher values) of the poles/zeros.

The plot of the log magnitude vs. input frequency is only half of the story. We also need to be able to plot the phase angle vs. input frequency on a log scale as well to complete the full Bode diagram.

For our original transfer function,

\[
H(jw) = \frac{Kz_1(jw/\omega_i + 1)}{jwp_1(jw/\omega_p + 1)}
\]

the cumulative phase angle associated with this function are given by

\[
\angle H(jw) = \angle K \angle z_1(jw/\omega_i + 1) - \angle jw \angle p_1(jw/\omega_p + 1)
\]

Then the cumulative phase angle as a function of the input frequency may be written as

\[
\angle H(jw) = \angle \left[ K + \angle z_1 + \frac{\angle jw}{\angle \omega_i} + 1 \right] - \angle \left[ \angle jw - \angle \omega_p - \frac{\angle jw}{\angle \omega_p + 1} \right]
\]

Once again, to show the phase plot of the Bode diagram, lines can be drawn for each of the different terms. Then the total effect may be found by superposition.

Effect of Constants on Phase:
A positive constant, \(K>0\), has no effect on phase. A negative constant, \(K<0\), will set up a phase shift of \(\pm 180^\circ\). (Remember real vs imaginary plots – a negative real number is at \(\pm 180^\circ\) relative to the origin)

Effect of Zeros at the origin on Phase Angle:
Zeros at the origin, \(s\), cause a constant \(+90^\circ\) degree shift for each zero.

\[
\angle TF \quad +90 \text{ deg}
\]

Effect of Poles at the origin on Phase Angle:
Poles at the origin, \(s^{-1}\), cause a constant \(-90^\circ\) degree shift for each pole.

\[
\angle TF \quad -90 \text{ deg}
\]
**Effect of Zeros not at the origin on Phase Angle:**

Zeros not at the origin, like \( 1 + \frac{j \omega}{z_1} \), have no phase shift for frequencies **much lower** than \( z_1 \), have a +45 deg shift at \( z_1 \), and have a +90 deg shift for frequencies **much higher** than \( z_1 \).

\[ \angle H \]
\[ \text{To draw the lines for this type of term, the transition from } 0^\circ \text{ to } +90^\circ \text{ is drawn over 2 decades, starting at } 0.1z_1 \text{ and ending at } 10z_1. \]

**Effect of Poles not at the origin on Phase Angle:**

Poles not at the origin, like \( \frac{1}{1 + \frac{j \omega}{p_1}} \), have no phase shift for frequencies **much lower** than \( p_1 \), have a -45 deg shift at \( p_1 \), and have a -90 deg shift for frequencies **much higher** than \( p_1 \).

\[ \angle TF \]
\[ \text{To draw the lines for this type of term, the transition from } 0^\circ \text{ to } -90^\circ \text{ is drawn over 2 decades, starting at } 0.1p_1 \text{ and ending at } 10p_1. \]

When drawing the phase angle shift for **not-at-the-origin zeros and poles**, first locate the critical frequency of the zero or pole. Then start the transition 1 decade before, following a slope of ±45°/decade. Continue the transition until reaching the frequency one decade past the critical frequency.

Now let’s complete the Bode Phase diagrams for the previous examples:
Example 1:
For the Transfer Function given, sketch the Bode diagram which shows how the phase of the system is affected by changing input frequency.

\[ TF = \frac{1}{2s + 100} = \frac{1/100}{\frac{s}{50} + 1} \]

![Bode diagram showing phase shift and magnitude response of the system.](image)
Example 2:
Repeat for the transfer function,

\[ 20 \log |TF| \]

\[ TF = \frac{5 \times 10^4 s}{s^2 + 505s + 2500} = \frac{20}{\left(\frac{s}{5} + 1\right)\left(\frac{s}{500} + 1\right)} \]
Example 2 Solution:
Repeat for the transfer function,

\[ 20 \log |TF| \]

\[ TF = \frac{5 \times 10^4 s}{s^2 + 505s + 2500} = \frac{20}{\frac{s}{5} + 1}\left(\frac{s}{500} + 1\right) \]
Example 3: Find the Bode log magnitude and phase angle plot for the transfer function,

\[ TF = \frac{200(s + 20)}{s(2s + 1)(s + 40)} = \frac{100 \left(\frac{s}{20} + 1\right)}{s\left(\frac{s}{0.5} + 1\right)\left(\frac{s}{40} + 1\right)} \]
Example 3: Find the Bode log magnitude and phase angle plot for the transfer function,

\[
TF = \frac{200(s + 20)}{s(2s + 1)(s + 40)} = \frac{100 \left(\frac{s}{20} + 1\right)}{s\left(\frac{s}{0.5} + 1\right)\left(\frac{s}{40} + 1\right)}
\]
Example 4:
Sketch the Bode plot (Magnitude and Phase Angle) for

\[
TF = \frac{100 \times 10^3 (s + 1)}{(s + 10)(s + 1000)}
\]

\[20 \log_{10} |TF|\]

Angle of \(TF\)
Example 4:
Sketch the Bode plot (Magnitude and Phase Angle) for

\[
TF = \frac{100 \times 10^3 (s + 1)}{(s + 10)(s + 1000)} = \frac{10 \left(\frac{s}{10} + 1\right)}{\left(\frac{s}{1000} + 1\right)}
\]

Therefore: \( K = 10 \) so \( 20 \log_{10}(10) = 20 \) db

One zero: \( z_1 = 1 \)

Two poles: \( p_1 = 10 \) and \( p_2 = 1000 \)

\[20 \log_{10}|TF|\]
Matlab can also be used to draw Bode plots:

Matlab (with the sketched Bode Plot superimposed on the actual plot)

$$TF = \frac{100 \times 10^3 (s + 1)}{(s + 10)(s + 1000)}$$

```matlab
w=logspace(-1,5,100);              %setup for x-axis
MagH=100000*sqrt(w.^2+1^2)./(sqrt(w.^2+10^2).*sqrt(w.^2+1000^2));
%transfer function
MagHdb=20*log10(MagH);         %transfer function converted to dB
PhaseHRad=atan(w/1)-atan(w/10)-atan(w/1000);   %phase done in radians
PhaseHDeg=PhaseHRad*180/pi;       %phase done in degrees
subplot(2,1,1)
semilogx(w,MagHdb,:b',x,y,'-b')       %semilog plot
xlabel('frequency [rad/s]'),ylabel('20 log10(|TF|) [db]'),grid %xaxis label
subplot(2,1,2)
semilogx(w,PhaseHDeg,:b',xAng,yAngDeg,-b')
xlabel('frequency [rad/s]'),ylabel('Phase Angle [deg]'),grid
```
Notice that the actual plot does not follow the sketched plot exactly. There is error between our sketched method and the actual Bode plot. How much error is expected?

Let’s look at an example of a zero, \( TF = (1 + \frac{s}{10}) \). Note, \( ?_{\text{critical}} = 10 \text{ rad/s} \)
The largest error that occurs on the Magnitude plot is right at the critical frequency. It is on the order of 3 dB.

The largest error that is shown on the Phase plot occurs at $0.1\, \omega_{\text{critical}}$ and $10\, \omega_{\text{critical}}$ (one decade above and below the critical frequency). Error at these points is about 6 degrees.

It’s understood that sketching the Bode diagrams will contain some error but this is generally considered acceptable practice.
To quickly sketch the graphs:

1. Determine the starting value: $|H(0)|$
2. Determine all critical frequencies (break frequencies). Start from the lowest value and draw the graphs as follows:

| Pole is negative | -20dB/dec | -45° |
| Pole is positive | -20dB/dec | +45° |
| Zero is negative  | +20dB/dec  | +45° |
| Zero is positive  | +20dB/dec  | -45° |

Add each value to the previous value.

Examples:

1. $H(s) = \frac{s + 1}{s + 10}$
   
   $|H(0)| = |0 + 1/0 + 10| = 1/10 = 0.1 \Rightarrow -20$dB

   Critical frequencies: zero@ -1 and pole@ -10

   ![Magnitude Plot](image)

   ![Phase Plot](image)

   The dotted line is a more accurate representation.

2. $H(s) = \frac{10(s - 1)}{(s - 3)(s - 10)}$

   $|H(0)| = |10*(-1)/(-3)*(-10)| = |1/3| = 1/3 \Rightarrow -10$dB

   Note that the angle of (-1/3 real value) is 180°

   critical frequencies: zero @ 1, pole@ 3 and 10

   ![Magnitude Plot](image)

   ![Phase Plot](image)