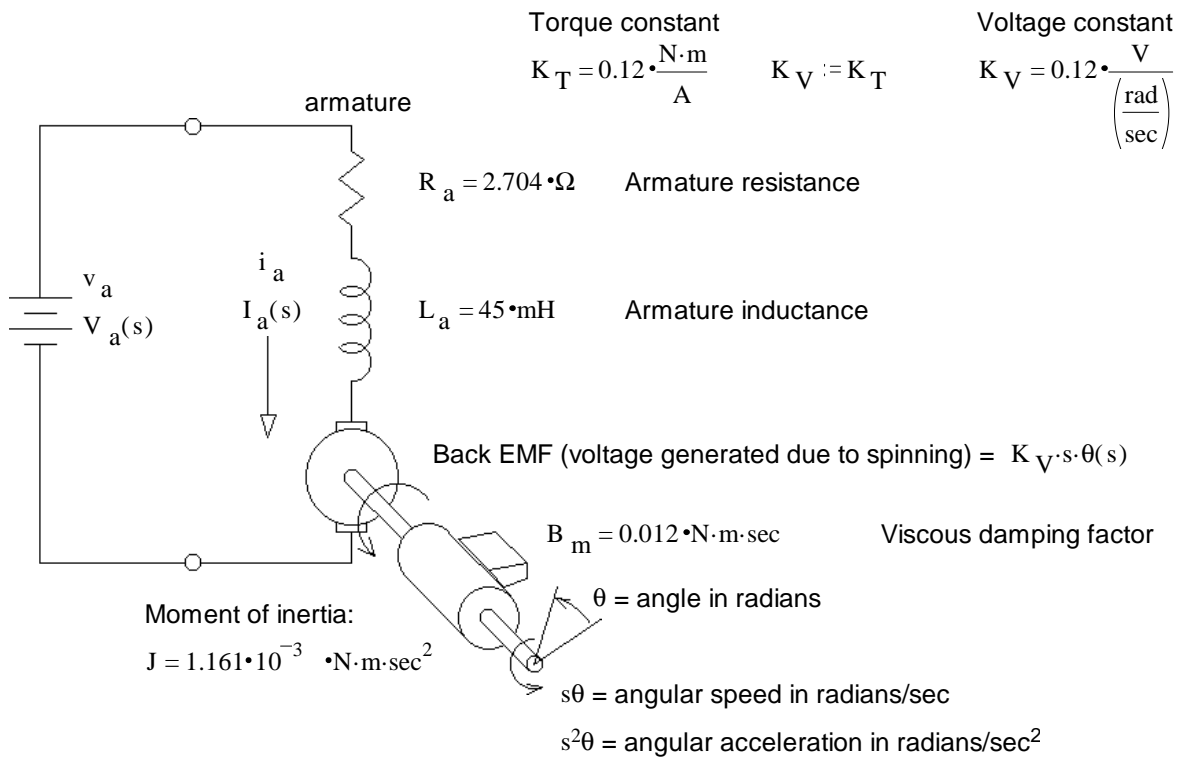


# ECE 1250 Appendix: Servo Calculations

For the DC permanent-magnet motor and gears taken together



$$\text{Torque: } T(s) = J \cdot (s^2 \cdot \theta(s)) + B_m \cdot (s \cdot \theta(s)) = K_T \cdot I_a(s) = K_T \cdot \frac{V_a(s) - K_V \cdot s \cdot \theta(s)}{R_a + L_a \cdot s}$$

$$J \cdot (s^2 \cdot \theta(s)) + B_m \cdot (s \cdot \theta(s)) = K_T \cdot \frac{V_a(s) - K_V \cdot s \cdot \theta(s)}{R_a + L_a \cdot s}$$

$$\left[ J \cdot (s^2 \cdot \theta(s)) + B_m \cdot (s \cdot \theta(s)) \right] \cdot (R_a + L_a \cdot s) = K_T \cdot (V_a(s) - K_V \cdot s \cdot \theta(s))$$

$$J \cdot s^2 \cdot \theta(s) \cdot R_a + J \cdot s^3 \cdot \theta(s) \cdot L_a + B_m \cdot s \cdot \theta(s) \cdot R_a + B_m \cdot s^2 \cdot \theta(s) \cdot L_a = K_T \cdot V_a(s) - K_T \cdot K_V \cdot s \cdot \theta(s)$$

$$J \cdot s^2 \cdot \theta(s) \cdot R_a + J \cdot s^3 \cdot \theta(s) \cdot L_a + B_m \cdot s \cdot \theta(s) \cdot R_a + B_m \cdot s^2 \cdot \theta(s) \cdot L_a + K_T \cdot K_V \cdot s \cdot \theta(s) = K_T \cdot V_a(s)$$

$$\left( J \cdot s^2 \cdot R_a + J \cdot s^3 \cdot L_a + B_m \cdot s \cdot R_a + B_m \cdot s^2 \cdot L_a + K_T \cdot K_V \cdot s \right) \cdot \theta(s) = K_T \cdot V_a(s)$$

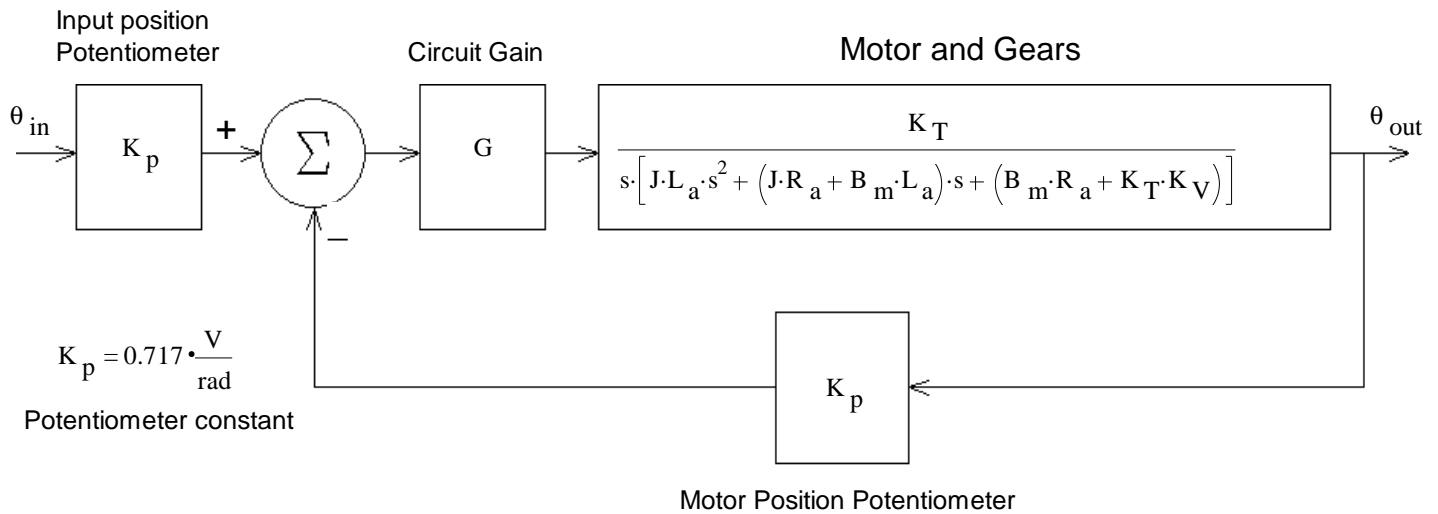
$$\left[ J \cdot L_a \cdot s^3 + (J \cdot R_a + B_m \cdot L_a) \cdot s^2 + (B_m \cdot R_a + K_T \cdot K_V) \cdot s \right] \cdot \theta(s) = K_T \cdot V_a(s)$$

$$\text{Motor transfer function: } \frac{\theta(s)}{V_a(s)} = \frac{K_T}{J \cdot L_a \cdot s^3 + (J \cdot R_a + B_m \cdot L_a) \cdot s^2 + (B_m \cdot R_a + K_T \cdot K_V) \cdot s}$$

$$= \frac{K_T}{s \cdot \left[ J \cdot L_a \cdot s^2 + (J \cdot R_a + B_m \cdot L_a) \cdot s + (B_m \cdot R_a + K_T \cdot K_V) \right]}$$

## ECE 1250 Servo Calculations, p2

The system block diagram



Overall System transfer function:

$$\frac{\theta_{\text{out}}(s)}{\theta_{\text{in}}(s)} = K_p \cdot \frac{G \cdot \frac{K_T}{s \cdot [J \cdot L_a \cdot s^2 + (J \cdot R_a + B_m \cdot L_a) \cdot s + (B_m \cdot R_a + K_T \cdot K_V)]}}{1 + K_p \cdot G \cdot \frac{K_T}{s \cdot [J \cdot L_a \cdot s^2 + (J \cdot R_a + B_m \cdot L_a) \cdot s + (B_m \cdot R_a + K_T \cdot K_V)]}}$$

$$\times \frac{s \cdot [J \cdot L_a \cdot s^2 + (J \cdot R_a + B_m \cdot L_a) \cdot s + (B_m \cdot R_a + K_T \cdot K_V)]}{s \cdot [J \cdot L_a \cdot s^2 + (J \cdot R_a + B_m \cdot L_a) \cdot s + (B_m \cdot R_a + K_T \cdot K_V)]}$$

$$\frac{\theta_{\text{out}}(s)}{\theta_{\text{in}}(s)} = \frac{G \cdot K_T \cdot K_p}{s \cdot [J \cdot L_a \cdot s^2 + (J \cdot R_a + B_m \cdot L_a) \cdot s + (B_m \cdot R_a + K_T \cdot K_V)] + K_p \cdot G \cdot K_T}$$

$$\frac{\theta_{\text{out}}(s)}{\theta_{\text{in}}(s)} = \frac{k \cdot K_T \cdot K_p}{J \cdot L_a \cdot s^3 + (J \cdot R_a + B_m \cdot L_a) \cdot s^2 + (B_m \cdot R_a + K_T \cdot K_V) \cdot s + K_p \cdot G \cdot K_T}$$

$$\frac{\theta_{\text{out}}(s)}{\theta_{\text{in}}(s)} = \frac{\frac{k \cdot K_T \cdot K_p}{J \cdot L_a}}{s^3 + \left( \frac{J \cdot R_a + B_m \cdot L_a}{J \cdot L_a} \right) \cdot s^2 + \left( \frac{B_m \cdot R_a + K_T \cdot K_V}{J \cdot L_a} \right) \cdot s + \frac{K_p \cdot G \cdot K_T}{J \cdot L_a}}$$

The characteristic equation:  $0 = s^3 + p \cdot s^2 + q \cdot s + r$

Where:

$$p := \frac{J \cdot R_a + B_m \cdot L_a}{J \cdot L_a} \quad q := \frac{B_m \cdot R_a + K_T \cdot K_V}{J \cdot L_a} \quad r := \frac{K_p \cdot G \cdot K_T}{J \cdot L_a}$$

### ECE 1250 Servo Calculations, p3

for Gain =  $G := 1.7$

$$r := \frac{K_p \cdot G \cdot K_T}{J \cdot L_a} \quad a := q - \frac{p^2}{3} \quad b := \frac{2 \cdot p^3}{27} - \frac{p \cdot q}{3} + r \quad A := \sqrt[3]{-\frac{b}{2} + \sqrt{\frac{(b)^2}{4} + \frac{a^3}{27}}} \quad B := \sqrt[3]{-\frac{b}{2} - \sqrt{\frac{(b)^2}{4} + \frac{a^3}{27}}}$$

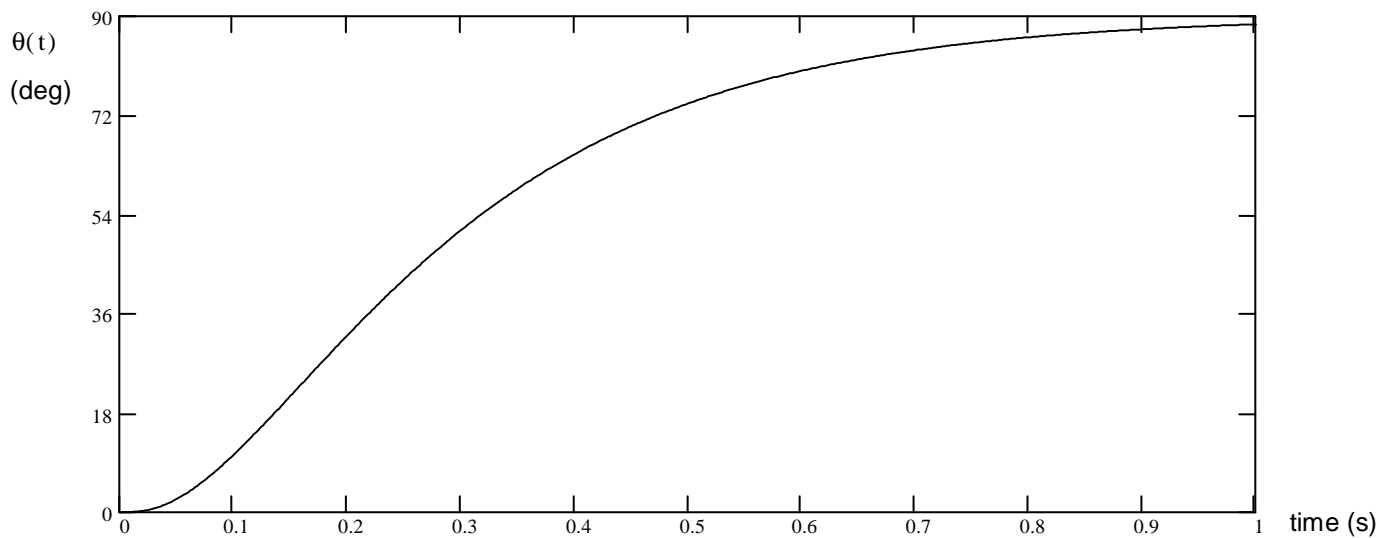
$$s_1 := (A + B) - \frac{p}{3} \quad s_1 = -4.83 \cdot \text{sec}^{-1} \quad s_2 := -\frac{A + B}{2} - \frac{A - B}{2} \cdot \sqrt{-3} - \frac{p}{3} \quad s_2 = -10.494 \cdot \text{sec}^{-1}$$

$$\theta_{\text{fin}} := 90 \cdot \text{deg} \quad s_3 := -\frac{A + B}{2} + \frac{A - B}{2} \cdot \sqrt{-3} - \frac{p}{3} \quad s_3 = -55.091 \cdot \text{sec}^{-1}$$

$$\begin{pmatrix} D \\ E \\ F \end{pmatrix} := \begin{bmatrix} 1 & 1 & 1 \\ s_1 \cdot \text{sec} & s_2 \cdot \text{sec} & s_3 \cdot \text{sec} \\ (s_1)^2 \cdot \text{sec}^2 & (s_2)^2 \cdot \text{sec}^2 & (s_3)^2 \cdot \text{sec}^2 \end{bmatrix}^{-1} \cdot \begin{bmatrix} -\theta_{\text{fin}} \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{pmatrix} D \\ E \\ F \end{pmatrix} = \begin{pmatrix} -3.19 \\ 1.655 \\ -0.036 \end{pmatrix}$$

$$\theta(t) := \theta_{\text{fin}} + D \cdot e^{s_1 t} + E \cdot e^{s_2 t} + F \cdot e^{s_3 t}$$



for Gain =  $G := 14$

$$r := \frac{K_p \cdot G \cdot K_T}{J \cdot L_a} \quad a := q - \frac{p^2}{3} \quad b := \frac{2 \cdot p^3}{27} - \frac{p \cdot q}{3} + r \quad A := \sqrt[3]{-\frac{b}{2} + \sqrt{\frac{(b)^2}{4} + \frac{a^3}{27}}} \quad B := \sqrt[3]{-\frac{b}{2} - \sqrt{\frac{(b)^2}{4} + \frac{a^3}{27}}}$$

$$s_1 := (A + B) - \frac{p}{3} \quad s_1 = -61.962 \cdot \text{sec}^{-1} \quad s_2 := -\frac{A + B}{2} - \frac{A - B}{2} \cdot \sqrt{-3} - \frac{p}{3} \quad s_2 = -4.227 - 18.796j \cdot \text{sec}^{-1}$$

$$\theta_{\text{fin}} := 90 \cdot \text{deg} \quad s_3 := -\frac{A + B}{2} + \frac{A - B}{2} \cdot \sqrt{-3} - \frac{p}{3} \quad s_3 = -4.227 + 18.796j \cdot \text{sec}^{-1}$$

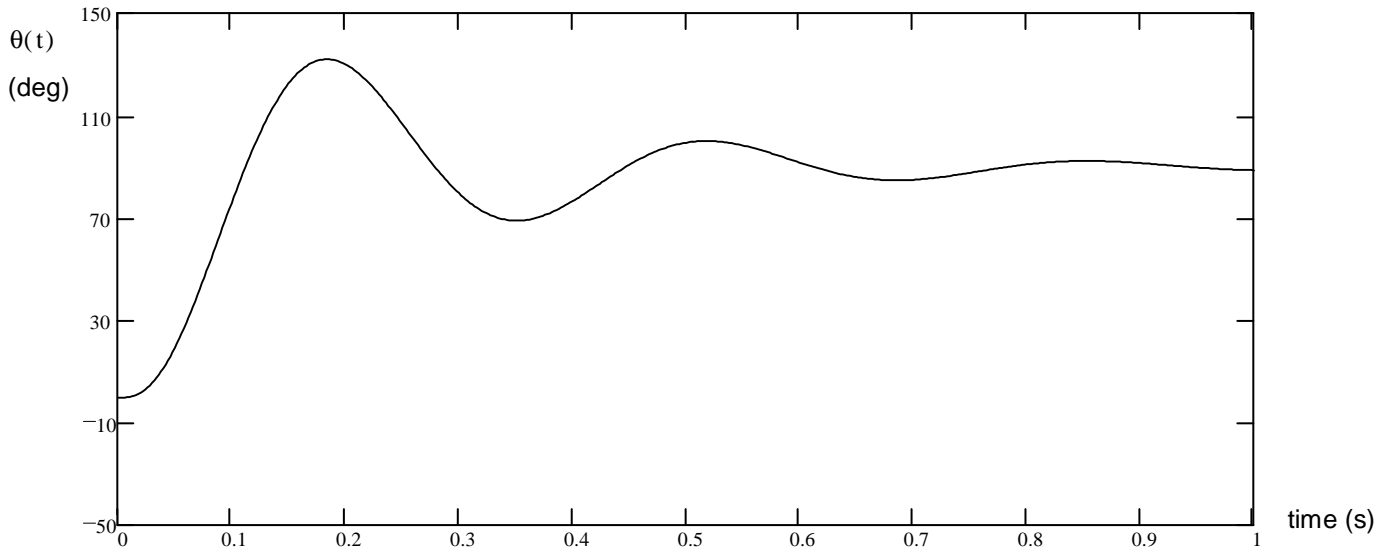
$$\begin{pmatrix} D \\ E \\ F \end{pmatrix} := \begin{bmatrix} 1 & 1 & 1 \\ s_1 \cdot \text{sec} & s_2 \cdot \text{sec} & s_3 \cdot \text{sec} \\ (s_1)^2 \cdot \text{sec}^2 & (s_2)^2 \cdot \text{sec}^2 & (s_3)^2 \cdot \text{sec}^2 \end{bmatrix}^{-1} \cdot \begin{bmatrix} -\theta_{\text{fin}} \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{pmatrix} D \\ E \\ F \end{pmatrix} = \begin{pmatrix} -0.158 \\ -0.706 - 0.419j \\ -0.706 + 0.419j \end{pmatrix}$$

$$\theta(t) := \theta_{\text{fin}} + D \cdot e^{s_1 t} + E \cdot e^{s_2 t} + F \cdot e^{s_3 t}$$

Plot on next page

# ECE 1250 Servo Calculations, p4



G = 14, Decaying Oscillation (ringing)

for Gain = G := 40

$$r := \frac{K_p \cdot G \cdot K_T}{J \cdot L_a} \quad a := q - \frac{p^2}{3} \quad b := \frac{2 \cdot p^3}{27} - \frac{p \cdot q}{3} + r \quad A := \sqrt[3]{-\frac{b}{2} + \sqrt{\frac{(b)^2}{4} + \frac{a^3}{27}}} \quad B := \sqrt[3]{-\frac{b}{2} - \sqrt{\frac{(b)^2}{4} + \frac{a^3}{27}}}$$

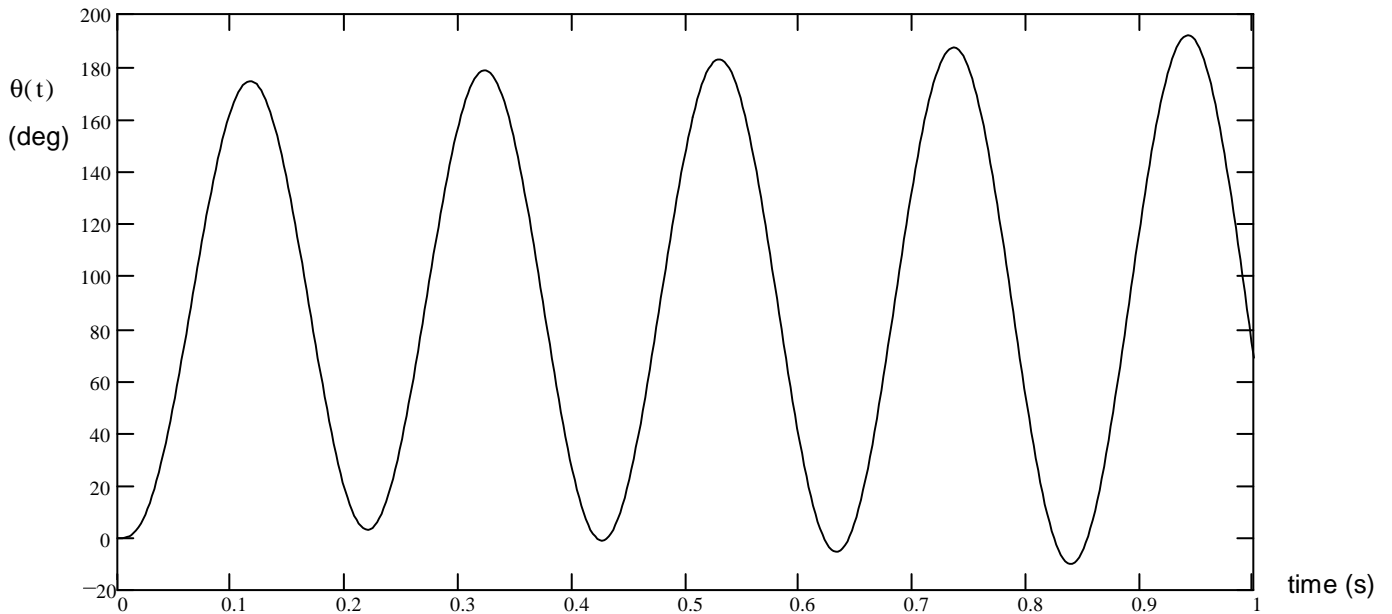
$$s_1 := (A + B) - \frac{p}{3} \quad s_1 = -70.87 \cdot \text{sec}^{-1} \quad s_2 := -\frac{A + B}{2} - \frac{A - B}{2} \cdot \sqrt{-3} - \frac{p}{3} \quad s_2 = 0.227 - 30.448j \cdot \text{sec}^{-1}$$

$$\theta_{\text{fin}} := 90 \cdot \text{deg} \quad s_3 := -\frac{A + B}{2} + \frac{A - B}{2} \cdot \sqrt{-3} - \frac{p}{3} \quad s_3 = 0.227 + 30.448j \cdot \text{sec}^{-1}$$

$$\begin{pmatrix} D \\ E \\ F \end{pmatrix} := \begin{bmatrix} 1 & 1 & 1 \\ s_1 \cdot \text{sec} & s_2 \cdot \text{sec} & s_3 \cdot \text{sec} \\ (s_1)^2 \cdot \text{sec}^2 & (s_2)^2 \cdot \text{sec}^2 & (s_3)^2 \cdot \text{sec}^2 \end{bmatrix}^{-1} \cdot \begin{bmatrix} -\theta_{\text{fin}} \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{pmatrix} D \\ E \\ F \end{pmatrix} = \begin{pmatrix} -0.243 \\ -0.664 - 0.278j \\ -0.664 + 0.278j \end{pmatrix}$$

$$\theta(t) := \theta_{\text{fin}} + D \cdot e^{s_1 \cdot t} + E \cdot e^{s_2 \cdot t} + F \cdot e^{s_3 \cdot t}$$



G = 40, Growing Oscillation