TOOL: A Butterworth filter is maximally flat in the passband and has a nearly linear phase response. The latter property means the shapes of waveforms are better-preserved. For this reason, the Butterworth filter is a good choice for filtering of speech or music. For example, when digitizing sound waveforms, anti-aliasing low-pass filters are used to eliminate high-frequency noise that masquerades as low frequencies.

The poles of a low-pass Butterworth filter lie on a circle of radius ω_c in the left half-plane in the s-domain. That means poles are located at the nth roots of -1.

$$\text{poles: } \omega_c e^{\frac{j(2k+n-1)\pi}{2n}} \quad \text{for } k = 1,2,3,4$$

For a 4th-order Butterworth low-pass filter, we have the following transfer function:

$$H(s) = \frac{V_o(s)}{V_i(s)} = \frac{\omega_c^2}{\left(s-\omega_c e^{j5\pi/8}\right)\left(s-\omega_c e^{-j5\pi/8}\right)} \cdot \frac{\omega_c^2}{\left(s-\omega_c e^{j7\pi/8}\right)\left(s-\omega_c e^{-j7\pi/8}\right)}$$

Alternate forms:

$$H(s) = \frac{V_o(s)}{V_i(s)} = \frac{\omega_c^2}{(s+\alpha_1+j\beta_1)(s+\alpha_1-j\beta)} \cdot \frac{\omega_c^2}{(s+\alpha_2+j\beta_2)(s+\alpha_2-j\beta_2)}$$

$$H(s) = \frac{\omega_c^2}{(s+\alpha_1)^2+\beta_1^2} \cdot \frac{\omega_c^2}{(s+\alpha_2)^2+\beta_2^2} = \frac{\omega_c^2}{(s+\alpha_1)^2+\omega_c^2-\alpha_1^2} \cdot \frac{\omega_c^2}{(s+\alpha_2)^2+\omega_c^2-\alpha_2^2}$$

$$\alpha_1 = \omega_c \cos(5\pi/8) = 0.383\omega_c$$

$$\beta_1 = \omega_c \sin(5\pi/8) = 0.924\omega_c$$

$$Q_1 = \frac{\omega_c}{2\alpha_1} \doteq 1.31$$

$$\alpha_2 = \omega_c \cos(7\pi/8) = 0.924\omega_c$$

$$\beta_2 = \omega_c \sin(7\pi/8) = 0.383\omega_c$$

$$Q_2 = \frac{\omega_c}{2\alpha_2} \doteq 0.54$$

REF: [1] Wikipedia, Butterworth filter, https://en.wikipedia.org/wiki/Butterworth_filter (accessed Nov. 8, 2020).