DERIV: The 2-pole low-pass filter magnitude response in terms of Q may be written as

$$|H(j\omega)| = \frac{1}{\left|\left(\frac{j\omega}{\omega_0}\right)^2 + \frac{1}{Q}\left(\frac{j\omega}{\omega_0}\right) + 1\right|}.$$

The response has a resonant peak near ω_0 for high Q. We wish to find the peak magnitude.

In CTool "FILTERS:BODE PLOTS:2-pole low-pass:PEAK RESPONSE FREQ DERIVATION", the frequency at the peak is shown to be at

$$\frac{\omega_{max}}{\omega_0} = \sqrt{1 - \frac{1}{2Q^2}}.$$

Substituting gives

$$|H(j\omega_{max})| = \frac{1}{\left|\frac{1}{2Q^2} - 1 + j\frac{1}{Q}\sqrt{1 - \frac{1}{2Q^2}} + 1\right|}$$

or

$$|H(j\omega_{max})| = \frac{1}{\sqrt{\left(\frac{1}{2Q^2}\right)^2 + \frac{1}{Q^2}\left(1 - \frac{1}{2Q^2}\right)}}.$$

Define $q = Q^2$ to simplify notation.

$$|H(j\omega_{max})| = \frac{1}{\sqrt{\left(\frac{1}{2q}\right)^2 + \frac{1}{q}\left(1 - \frac{1}{2q}\right)}}$$

or

$$|H(j\omega_{max})| = \frac{1}{\sqrt{\frac{1}{4q^2} + \frac{1}{q} - \frac{1}{2q^2}}} = \frac{1}{\sqrt{\frac{1}{q} - \frac{1}{4q^2}}}$$

or

$$|H(j\omega_{max})| = \frac{1}{\sqrt{\frac{1}{Q^2} - \frac{1}{4Q^4}}} = \frac{Q}{\sqrt{1 - \frac{1}{4Q^2}}}$$

For large Q (in practice, $Q > 2$ will suffice), may use an approximation for the square root.

$$|H_{max}| \approx \frac{Q}{1 - \frac{1}{8Q^2}}$$

Then we may use an approximation for the denominator.

$$1 - x \approx \frac{1}{1 + x}$$

This gives a simple formula that ultimately simplifies to just Q:

$$|H_{max}| \approx Q\left(1 + \frac{1}{8Q^2}\right) = Q + \frac{1}{8Q} \approx Q$$

The table below shows that, if approximate the peak as Q, we get less than 3% error for $Q = 2$, and we get errors less than 1% for $Q \geq 4$. There is hardly a peak for $Q < 1$.

$\zeta = \frac{1}{2Q}$	Q	$\frac{\omega_{max}}{\omega_0} = \sqrt{1 - \frac{1}{2Q^2}}$	$\lvert H_{max}\rvert$	% approx err for $\lvert H_{max}\rvert = Q$	$\lvert H_{max}\rvert$ in dB	Q in dB
0.667	0.75	0.333	1.01	25.5	0.05	-2.50
0.625	0.8	0.468	1.02	21.9	0.21	-1.94
0.556	0.9	0.619	1.08	16.9	0.69	-0.92
0.500	1	0.707	1.15	13.4	1.25	0.00
0.250	2	0.935	2.07	3.2	6.30	6.02
0.167	3	0.972	3.04	1.4	9.66	9.54
0.125	4	0.984	4.03	0.8	12.11	12.04
0.100	5	0.990	5.03	0.5	14.02	13.98

$\zeta = \frac{1}{2Q}$	Q	$\frac{\omega_{max}}{\omega_0}$	$\lvert H_{max} \rvert$	% approx err	$\lvert H_{max} \rvert$ in dB	Q in dB
0.010	50.00	1.000	50.00	0.0	33.98	33.98
0.020	25.00	1.000	25.01	0.0	27.96	27.96
0.050	10.00	0.997	10.01	0.1	20.01	20.00
0.100	5.00	0.990	5.03	0.5	14.02	13.98
0.200	2.50	0.959	2.55	2.0	8.14	7.96
0.300	1.67	0.906	1.75	4.6	4.85	4.44
0.400	1.25	0.825	1.36	8.3	2.70	1.94
0.500	1.00	0.707	1.15	13.4	1.25	0.00
0.600	0.83	0.529	1.04	20.0	0.35	-1.58
0.700	0.71	0.141	1.00	28.6	0.00	-2.92

REF: Wolfram Alpha (Taylor series)
https://www.wolframalpha.com/input/?i=taylor+series+%281-x%29%5E1%2F2 accessed 11/15/2020.