DERIV: The 2-pole low-pass filter magnitude response

$$|H(j\omega)| = \frac{1}{\left|\left(\frac{j\omega}{\omega_0}\right)^2 + \frac{1}{Q}\left(\frac{j\omega}{\omega_0}\right) + 1\right|}$$

has a resonant peak for high Q.

The resonant peak, $|H|_{\max}$ is at a different frequency, ω_m, close to but different than the resonant frequency, ω_0. To find ω_m, we use the idea from calculus that the maximum occurs where the derivative of a function is zero.

$$\frac{d}{d\omega}|H(j\omega)| = 0$$

To simplify the calculation, we use magnitude squared. The peak of the magnitude squared will occur at the same frequency as the peak of the magnitude.

$$\frac{d}{d\omega}|H(j\omega)|^2 = 0$$

or

$$0 = \frac{d}{d\omega}|H(j\omega)|^2 = \frac{d}{d\omega}\frac{1}{\left|\left(\frac{j\omega}{\omega_0}\right)^2 + \frac{1}{Q}\left(\frac{j\omega}{\omega_0}\right) + 1\right|^2}$$

or, taking the sum of the squares of the imaginary and real parts,

$$0 = \frac{d}{d\omega}\frac{1}{\left[-\left(\frac{\omega}{\omega_0}\right)^2 + 1\right]^2 + \frac{1}{Q^2}\left(\frac{\omega}{\omega_0}\right)^2}$$

We define a convenient, normalized, squared frequency term for the remainder of the calculation.

$$w \equiv \left(\frac{\omega}{\omega_0}\right)^2$$

or

$$0 = \frac{d}{d\omega} \frac{1}{(-\mathrm{w}+1)^2 + \frac{1}{Q^2}\mathrm{w}}$$

or

$$0 = \frac{d}{d\omega}\left[(-\mathrm{w}+1)^2 + \frac{1}{Q^2}\mathrm{w}\right]^{-1}$$

or

$$0 = -\left[(-\mathrm{w}+1)^2 + \frac{1}{Q^2}\mathrm{w}\right]^{-2} \frac{d}{d\omega}\left[(-\mathrm{w}+1)^2 + \frac{1}{Q^2}\mathrm{w}\right]$$

The denominator term must *not* be zero, so the term to the -2 power may be ignored.

$$0 = \frac{d}{d\omega}\left[(-\mathrm{w}+1)^2 + \frac{1}{Q^2}\mathrm{w}\right] = 2(-\mathrm{w}+1)(-1) + \frac{1}{Q^2}$$

or

$$\mathrm{w} - 1 = -\frac{1}{2Q^2}$$

or

$$\mathrm{w} = 1 - \frac{1}{2Q^2}$$

or

$$\left(\frac{\omega_{\max}}{\omega_0}\right)^2 = 1 - \frac{1}{2Q^2}$$

or, for the exact result,

$$\frac{\omega_{\max}}{\omega_0} = \sqrt{1 - \frac{1}{2Q^2}}$$

For small x, we have the following Taylor series expansion.

$$\sqrt{1-x} = 1 - \frac{x}{2} + O(x^2)$$

Thus, for high Q, we have the following approximation.

$$\frac{\omega_{\max}}{\omega_0} \approx 1 - \frac{1}{4Q^2}$$

For $Q > 4$, the approximation is extremely accurate. However, $\omega_{\max}$ is approximately ω_0 for $Q > 4$ anyway, so the approximation has limited value. The following table lists some values in the useful range of Q. Note that there is no peak in the frequency response for $Q < 1/\sqrt{2} \doteq 0.707$.

$\zeta = \frac{1}{2Q}$	Q	$\frac{\omega_{\max}}{\omega_0} = \sqrt{1 - \frac{1}{2Q^2}}$	$\log_{10}\left(\frac{\omega_{\max}}{\omega_0}\right)$	$\frac{\omega_{\max}}{\omega_0} \approx 1 - \frac{1}{4Q^2}$	% approx err
0.667	0.75	0.333	-0.48	0.56	66.67
0.625	0.8	0.468	-0.33	0.61	30.29
0.556	0.9	0.619	-0.21	0.69	11.75
0.500	1	0.707	-0.15	0.75	6.07
0.250	2	0.935	-0.03	0.94	0.22
0.167	3	0.972	-0.01	0.97	0.04
0.125	4	0.984	-0.01	0.98	0.01
0.100	5	0.990	0.00	0.99	0.01
0.100	5.00	0.990	0.00	0.99	0.01
0.200	2.50	0.959	-0.02	0.96	0.09
0.300	1.67	0.906	-0.04	0.91	0.49
0.400	1.25	0.825	-0.08	0.84	1.86
0.500	1.00	0.707	-0.15	0.75	6.07
0.600	0.83	0.529	-0.28	0.64	20.95
0.700	0.71	0.141	-0.85	0.51	260.62

REF: Wolfram Alpha (Taylor series)
https://www.wolframalpha.com/input/?i=taylor+series+%281-x%29%5E1%2F2 accessed 11/15/2020.