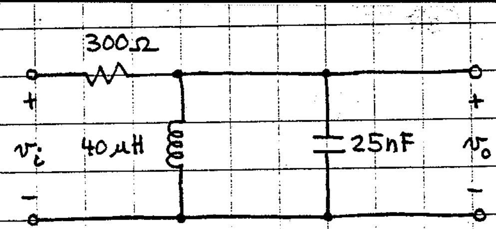
Ref: Nilsson & Riedel, Rev 6, Prob 14.12



For above bandpass filter find:
a) ω_0 b) f_0 c) Q d) ω_{c1} e) f_{c1} f) ω_{c2} g) f_{c2} h) β

ans: a) $\omega_0 = 1\text{ M rad/s}$ d) $\omega_{c1} = 936\text{ krad/s}$ g) $170\text{ kHz} = f_{c2}$
b) $f_0 = 159\text{ kHz}$ e) $f_{c1} = 149\text{ kHz}$ h) $\beta = 21.2\text{ kHz}$
c) $Q = 7.5$ f) $\omega_{c2} = 1.07\text{ Mrad/s}$

sol'n: a) Derive the transfer function, $H(s=j\omega)$, and determine definitions of ω_0 and Q (for the sake of pedagogy).

We have a V-divider circuit:

$$V_o = V_i \cdot \frac{z_L \| z_C}{z_L \| z_C + R} \qquad z_L \| z_C = \frac{j\omega L \cdot \frac{1}{j\omega C}}{j\omega L + \frac{1}{j\omega C}} = \frac{\frac{L}{C}}{j\left(\omega L - \frac{1}{\omega C}\right)}$$

$$H(j\omega) \equiv \frac{V_o}{V_i} = \frac{z_L \| z_C}{z_L \| z_C + R} = \frac{\frac{L/C}{j(\omega L - 1/\omega C)}}{\frac{L/C}{j(\omega L - 1/\omega C)} + R}$$

$$H(j\omega) = \frac{L/C}{L/C + Rj(\omega L - 1/\omega C)} = \frac{1}{1 + jR(\omega C - 1/\omega L)}$$

The center frequency, ω_0, is defined as: (for bandpass)

1) The frequency where $H(j\omega)$ is purely Real

or 2) " " " $\angle H(j\omega) = 0$ (i.e. same as (1))

or 3) " " " $|H(j\omega)|$ is maximum.

The third definition captures the concept of finding the ω that maximizes the magnitude gain.

(1) and (2) are equivalent to (3) by the following argument:

1) The $\max\limits_{\omega}$ of $|H(j\omega)|$ occurs where $|H(j\omega)|^2$ is max.

2) $$|H(j\omega)|^2 = \frac{1}{\underset{\text{denom}}{Re[H(j\omega)]^2} + \underset{\text{denom}}{Im[H(j\omega)]^2}} \quad \text{(for } H(j\omega) \text{ in this prob).}$$

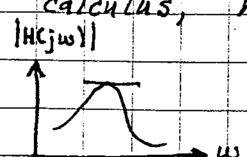
$$= \frac{1}{\underbrace{1}_{Re^2} + \underbrace{R^2(\omega C - 1/\omega L)^2}_{Im^2}}$$

3) Since ω only appears in Im^2 term, and since $Im^2 \geq 0$ (squared real is ≥ 0), it follows that the smallest denominator for $H(j\omega)$ occurs when $Im^2 = 0$.

$$\therefore \omega_0 C = \frac{1}{\omega_0 L} \quad \text{or} \quad \omega_0^2 = \frac{1}{LC} \quad \text{or} \quad \omega_0 = \frac{1}{\sqrt{LC}}$$

This works for $H(j\omega)$ of a certain form (that we always obtain for bandpass and bandreject filters). Suppose, however, that we were faced with the problem of finding $|H(j\omega)|$ max over ω for some other $H(j\omega)$. What might we do? We illustrate some useful tricks in the following derivation, (which is still tailored to the $H(j\omega)$ for this problem).

From calculus, $\max\limits_{\omega} |H(j\omega)|$ occurs when $\frac{d}{d\omega}|H(j\omega)| = 0$;



i.e. slope $= 0$ at max.

Unfortunately $\frac{d}{d\omega}|H(j\omega)|$ is messy; it has a $\frac{1}{\sqrt{\ }}$ form.

To simplify matters, we observe that $\max_\omega |H(j\omega)|$ and $\max_\omega |H(j\omega)|^2$ occur at the same value of ω.

$\therefore$ we solve $\frac{d}{d\omega}|H(j\omega)|^2 = 0$.

To simplify matters further, we write $|H(j\omega)|^2 = \frac{1}{G(j\omega)}$

where $G(j\omega) \equiv$ denominator of $|H(j\omega)|^2 \equiv 1 + R^2(\omega C - 1/\omega L)^2$.

$\therefore$ we solve $\frac{d}{d\omega}\frac{1}{G(j\omega)} = 0$

But this means, by the chain rule, that we have:

$$\frac{d}{d\omega}\frac{1}{G(j\omega)} = \frac{-1}{G(j\omega)^2}\frac{d}{d\omega}G(j\omega) = 0$$

So long as we avoid ω's for which $G(j\omega)^2 = \infty$ (namely $\omega = 0$ and $\omega = \infty$) we need only solve

$$\frac{d}{d\omega}G(j\omega) = 0 \qquad \left(\text{i.e. either } \frac{-1}{G(j\omega)^2} = 0 \text{ or } \frac{d}{d\omega}G(j\omega) = 0\right).$$

$$\therefore 0 = \frac{d}{d\omega}G(j\omega) = \frac{d}{d\omega}\left[1 + R^2(\omega C - 1/\omega L)^2\right] = R^2 2(\omega C - 1/\omega L) \cdot \frac{d}{d\omega}(\omega C - 1/\omega L)$$

or $$0 = \cancel{R^2} \cdot \cancel{2}\underbrace{(\omega C - 1/\omega L)}_{=0}\ \text{or}\ \underbrace{(C + 1/\omega^2 L)}_{=0}$$

Since $C, L, \omega \geq 0$ we can't have $C + 1/\omega^2 L = 0$.

$\therefore \omega_0 C - 1/\omega_0 L = 0$ or $\omega_0^2 = 1/LC$ or $\omega_0 = 1/\sqrt{LC}$.

Plug in #'s: $\omega_0 = \dfrac{1}{\sqrt{40\mu \cdot 25n}} = \dfrac{1}{\sqrt{1k\,\mu\,n}} = 1M$ rad/s

b) $\omega = 2\pi f \Rightarrow f_0 = \dfrac{\omega}{2\pi} = \dfrac{1M/s}{2\pi} = 159k/s$ or 159 kHz

c) $Q \equiv \dfrac{f_0}{f_{c_1} - f_2} \equiv \dfrac{\text{center freq}}{\text{higher cutoff freq} - \text{lower cutoff freq}}$

To find the Q, we could use book formulas, but we'll derive f_{c_1} and f_{c_2} here.

By definition $\omega_{c_1}, \omega_{c_2}$ are the frequencies where $|H(j\omega)| = \frac{1}{\sqrt{2}} |H(j\omega)|_{max}$.

Since $\max\limits_{\omega} |H(j\omega)| = 1$ we have:

$$|H(j\omega_c)| = \frac{1}{\sqrt{2}} \quad \text{or} \quad |H(j\omega_c)|^2 = \frac{1}{2}$$

$$\text{or} \quad \frac{1}{|H(j\omega_c)|^2} = 2 \quad \text{or} \quad 1 + R^2(\omega C - 1/\omega L)^2 = 2$$

$$\text{or} \quad R^2(\omega C - 1/\omega L)^2 = 1$$

We take the square root of both sides, and we must allow for a + or − square root

$$R(\omega C - 1/\omega L) = \pm 1$$

Multiply both sides by $\dfrac{\omega}{RC}$:

$$\omega^2 - \frac{1}{LC} = \pm \frac{\omega}{RC} \quad \text{or} \quad \omega^2 \pm \frac{\omega}{RC} - \frac{1}{LC} = 0$$

Because of the $\pm \dfrac{\omega}{RC}$ term, we have two quadration equations — each with two roots:

$$\omega_c = \pm \frac{1}{2RC} \pm \sqrt{\left(\frac{1}{2RC}\right)^2 + \frac{1}{LC}}$$ Are there four solutions? No!

Since $\omega_c \geq 0$, there are only two solutions.

Since $\sqrt{\left(\frac{1}{2RC}\right)^2 + \frac{1}{LC}} \geq \frac{1}{2RC}$ the two positive solutions are:

$$\omega_{c1} = -\frac{1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 + \frac{1}{LC}} \qquad \frac{1}{2RC} = \frac{1}{2\cdot 300\cdot 25n} = 66.7k/s$$

$$\omega_{c2} = \frac{1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 + \frac{1}{LC}} \qquad \frac{1}{LC} = (1M/s)^2$$

Now we may compute β and Q:

$$\beta = \omega_{c2} - \omega_{c1} = \frac{1}{RC}$$

$$Q = \frac{\omega_0}{\beta} = \frac{\frac{1}{\sqrt{LC}}}{\frac{1}{RC}} = \frac{RC}{\sqrt{LC}} = R\sqrt{\frac{C}{L}}$$

Now we plug in #'s:

$$Q = 300\cdot\sqrt{\frac{25n}{40\mu}} = 7.5$$

d) $$\omega_{c1} = -66.7k + \sqrt{(\underbrace{66.7k}_{\text{small re } (1M)^2})^2 + (1M)^2}\,/s = -66.7k + 1M = 933\ k/sec$$

Note: Ignoring the 66.7k inside the $\sqrt{\ }$ results in an error of 3 k/sec in our answer. The correct value is 936 k/sec. The source of this error is also attributable to taking the difference of two values. Any errors in terms we take the difference of gets magnified. For example 1.0001 M − 1.0000 M = 100, whereas 1.00011 M − 1.00000 M = 110. A tiny change in the first quantity, (i.e. a 0.001 % change) results in a 10% change in the outcome.

e) $f_{c1} = \dfrac{\omega_{c1}}{2\pi} = \dfrac{936\,k/s}{2\pi} = 149\,k\,Hz$

f) $\omega_{c2} = 66.7k/s + \sqrt{(\underbrace{66.7k/s}_{\text{small}})^2 + (1M/s)^2} = 1.07\,M/s$

g) $f_{c2} = \dfrac{\omega_{c2}}{2\pi} = \dfrac{149\,kHz}{2\pi} = 170\,kHz$

h) $\beta = \dfrac{1}{2\pi \cdot RC}\,Hz = \dfrac{1}{2\pi \cdot 300 \cdot 25n}\,Hz = 21.2\,kHz$

Note: If we use $\dfrac{\omega_{c2} - \omega_{c1}}{2\pi} = \dfrac{1.07M - 933k}{2\pi} = 21.8\,k\,Hz$,

a less accurate value. Again, problems arise from taking the (small) difference of two large values.