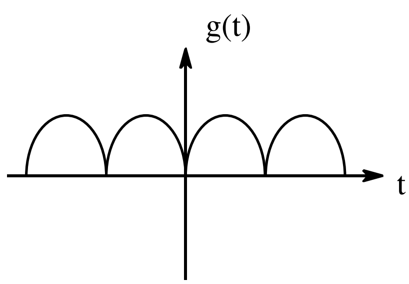
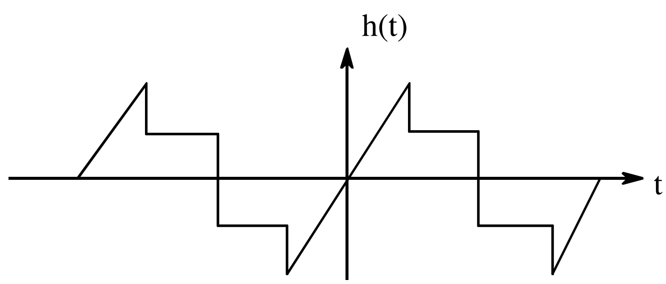
EX: Fill in the following table for the functions shown below.

	g(t)		h(t)	
	True	False	True	False
The function is odd				
The function is even				
The function has shift-flip symmetry				
The function has quarter-wave symmetry				
$a_v = 0$ (DC offset)				
All the a_k are zero				
All b_k are zero for even-numbered subscripts				





ANS:

	g(t)		h(t)	
	True	False	True	False
The function is odd		√1	√8	
The function is even	√2			√9
The function has shift-flip symmetry		√3		√10
The function has quarter-wave symmetry		√4		√11
$a_v = 0$ (DC offset)		√5	√12	
All the a_k are zero		√6	√13	
All b_k are zero for even-numbered subscripts	√7			√14

SOL'N: Answers are explained per superscript number.

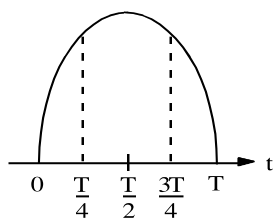
[1]g(t) is not odd because it is <u>not</u> equal to the original g(t) after being flipped around the vertical and horizontal axes.

[2]g(t) is even because it <u>is</u> symmetrical (and periodic, of course) around the vertical axis. Cosines are also even functions and so constitute the terms of Fourier series for even functions.

[3]Shift-flip symmetry means the function is equal to a copy of itself that is shifted one-half cycle to the right and flipped upside down. A cycle for g(t) is the width of one hump. If we shift g(t) to the right by half the width of a hump and then flip it upside down, then we obtain a function that is always negative. Thus, it is clearly <u>not</u> equal to the original g(t).

[4]Quarter-wave symmetry means the function has shift-flip symmetry and is symmetric to the left and right around the point at T/4 as well as to the left and right around the point at 3T/4. (Imagine placing a vertical axis at T/4 or 3T/4 and looking for mirror-image symmetry around that axis.) The period of g(t)

is one lobe, and we do not have mirror-image symmetry around T/4 and/or 3T/4:



[5]We have a zero DC offset only if g(t) has equal area above and below the horizontal axis. We may think of g(t) as being made of butter that we smooth out until it is perfectly flat. If the flat height is nonzero, then the DC offset is not zero.

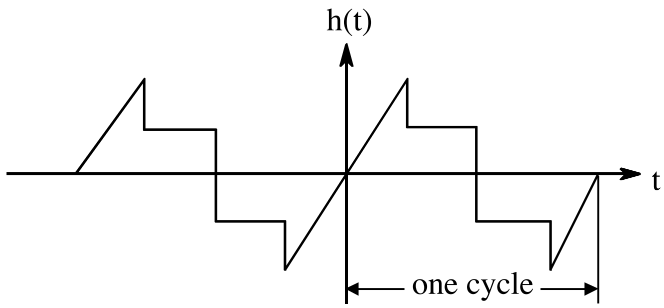
[6]The a_k coefficients are for cosine terms. Since g(t) is even (and nonzero) and cosine terms are even functions, we must have some cosine terms. (The sum of even functions is an even function.) Thus, the a_k are not all zero.

[7]The b_k coefficients for even-numbered subscripts are for sine terms with an even number of cycles per period of g(t). Since g(t) is even, we have only cosine terms, and all a_k are zero whether k is even or odd.

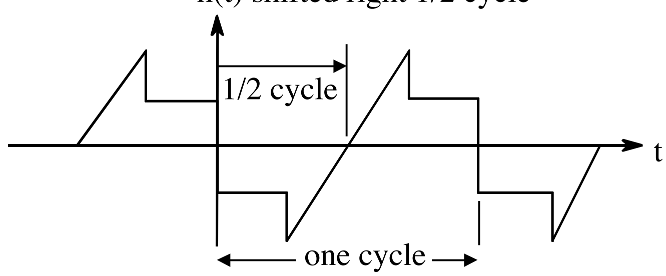
[8]h(t) is odd because is equal to the original g(t) after being flipped around the vertical and horizontal axes.

[9]h(t) is not symmetrical around the vertical axis. Thus, it is not even.

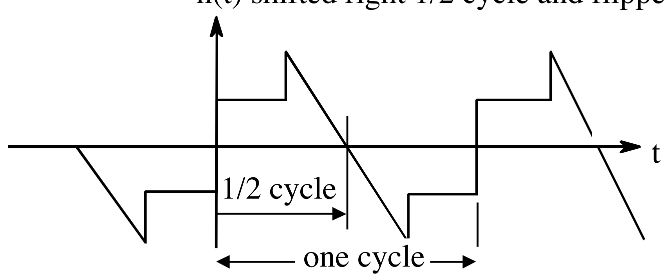
[10]Shift-flip symmetry means the function is equal to a copy of itself that is shifted one-half cycle to the right and flipped upside down. The following figures show h(t) shifted right by 1/2 cycle and then flipped upside down:



h(t) shifted right 1/2 cycle

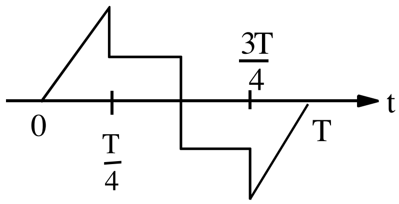


h(t) shifted right 1/2 cycle and flipped vertically



The last function is <u>not</u> the same as the original h(t). Thus h(t) does not have shift flip symmetry.

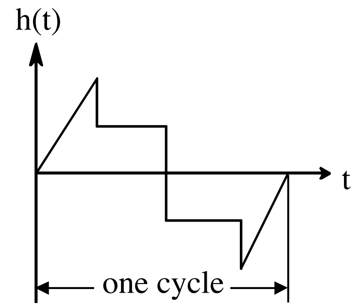
[11]Clearly, h(t) is not symmetrical about the quarter-wave points:

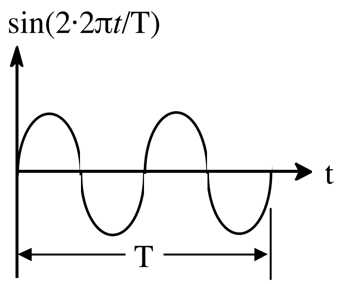


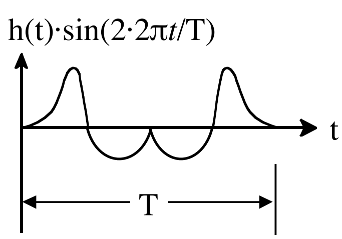
[12]We have a zero DC offset if h(t) has equal area above and below the horizontal axis. We may think of h(t) as being made of butter that we smooth out until it is perfectly flat. Clearly, the area under h(t) above the horizontal will exactly fill the area carved out by h(t) below the horizontal axis. Thus, the height after flattening is zero, and the DC offset is zero.

[13]The a_k coefficients are for cosine terms. Since h(t) is an odd function and sines are odd functions, we will have only sine terms. Thus, the a_k are all zero.

[14]The b_k coefficients for even-numbered subscripts are for sine terms with an even number of cycles per period of h(t). The figures below show h(t), sin(2·2πt/T) where T is the period, and h(t)sin(2·2πt/T). The coefficient, b_2, is equal to 2/T times the area under (i.e., integral of) one period of h(t)sin(2·2πt/T). This area appears to be zero, but the strange shapes prevent a definite conclusion from visual inspection alone.







Thus, we tackle the problem mathematically.

$$b_k = \frac{2}{T}\int_0^T h(t)\sin\left(k2\pi\frac{t}{T}\right)dt$$

Exploiting symmetry around T/2 for *k* even, we have

$$b_k = 2\cdot\left[\frac{2}{T}\int_0^{T/4}\frac{4t}{T}\sin\left(k2\pi\frac{t}{T}\right)dt + \frac{2}{T}\int_{T/4}^{T/2}\frac{1}{2}\sin\left(k2\pi\frac{t}{T}\right)dt\right].$$

From integral tables or a calculator, we have

$$\int x \sin(ax)dx = \frac{1}{a^2}\sin(ax) - \frac{x}{a}\cos(ax).$$

We may also assume that T = 1 without affecting our answer.

$$b_k = 4\left[4\int_0^{1/4} t\sin(k2\pi t)dt + \frac{1}{2}\int_{1/4}^{1/2}\sin(k2\pi t)dt\right]$$

$$b_k = 16\left[\frac{1}{(k2\pi)^2}\sin(k2\pi t) - \frac{t}{k2\pi}\cos(k2\pi t)\right]\Bigg|_0^{1/4} - 2\frac{1}{k2\pi}\cos(k2\pi t)\Bigg|_{1/4}^{1/2}$$

For k even and t = 0, $k2\pi t = 0$ and $\sin(k2\pi t) = 0$.

For k even and t = 1/4, $k2\pi t = \text{integer}\cdot\pi$ and $\sin(k2\pi t) = 0$.

For t = 0, $t\cdot\cos(k2\pi t) = 0$.

Thus, we have

$$b_k = 16\left[-\frac{t}{k2\pi}\cos(k2\pi t)\right]\Bigg|^{1/4} - 2\frac{1}{k2\pi}\cos(k2\pi t)\Bigg|_{1/4}^{1/2}$$

$$b_k = -\frac{4}{k2\pi}\cos(k2\pi/4) - 2\frac{1}{k2\pi}\cos(k2\pi/2) + 2\frac{1}{k2\pi}\cos(k2\pi/4)$$

$$b_k = -\frac{2}{k2\pi}\cos(k2\pi/4) - 2\frac{1}{k2\pi}\cos(k2\pi/2)$$

For $k = 2$, we have

$$b_2 = -\frac{1}{2\pi}\cos(\pi) - \frac{1}{2\pi}\cos(2\pi) = -\frac{1}{2\pi}(-1+1) = 0$$

So it seems that $b_k = 0$ might be true for all k even. Considering $k = 4$, however, we have

$$b_4 = -\frac{1}{4\pi}\cos(2\pi) - \frac{1}{4\pi}\cos(4\pi) = -\frac{1}{4\pi}(1+1) = -\frac{1}{2\pi}.$$

It follows that b_4 is not zero, and the b_k for k even are not all zero. Note that this problem exhibited an unusual symmetry that made the first even numbered b coefficient zero. Sometimes, the math is necessary. The standard types of symmetry and the pictures that go with them, however, tend to give results that are more obvious.

Note: In this problem, we could actually determine that b_2 is zero by observing that the initial sloped part of h(t) is symmetric around its center point located at T/8. The first hump in sin(2·2πt/T) is also symmetric around T/8. The height of h(t) at T/8 is equal to the height of the flat segment that follows. As we move to the left and right of T/8, we will multiply sin(2·2πt/T) by values that are equally above and below the height of h(t) at T/8. Thus, when we compute the integral of (i.e., area under) h(t)·sin(2·2πt/T) from 0 to T/4, we will have exactly the same value as we have for the integral of h(T/8)·sin(2·2πt/T) from 0 to T/4.

In other words, we get the same answer as we would get if we used a constant value for h(t), and that constant value is exactly the height of the flat segment that follows the initial sloped part of h(t). By symmetry, we conclude that we may replace the entire h(t) by the constant value h(T/8). Now b_2 is seen to be proportional to the integral of a constant times sin(2·2πt/T). The integral of any sinusoid of an integer number of cycles is zero, however, so we conclude that b_2 is zero.