Approximation Theory –

21 May 1990 Stone - Weierstrass Theorem Examples — Exponentials

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ex: Sums of exponentials satisfy the Stone - Weierstrass thm:

$$\tilde{\mathcal{F}} = \left\{ f(x) : f(x) = \sum_{n=0}^{\infty} a_n e^{nx} , \quad a_n \in \mathbb{R} \right\}$$

I) Identity function $f(x) = 1 \cdot e^{0x} = 1 \in \tilde{\mathcal{F}}$ ✓

II) Separability $f(x_1) = 1 \cdot e^{1x_1} \neq 1 \cdot e^{1x_2} = f(x_2)$ if $x_1 \neq x_2$ ✓

III) Algebraic Closure $f(x) = \sum_{n=0}^{\infty} a_n e^{nx} \quad g(x) = \sum_{m=0}^{\infty} b_m e^{mx}$

$$af(x) + bg(x) = \sum_{n=0}^{\infty} (aa_n + bb_n) e^{nx} \in \tilde{\mathcal{F}} \checkmark$$

$$f(x)\,g(x) = \sum_{k=0}^{\infty} \Big(\sum_{n+m=k} a_n b_m \Big) e^{kx}$$

$$= \sum_{k=0}^{\infty} c_k e^{kx} \in \tilde{\mathcal{F}} \text{ where } c_k = \sum_{n+m=k} a_n b_m$$

ex: *Exponentials with real valued multipliers but no summation do not satisfy S-W thm:*

$$\tilde{\mathcal{F}} = \left\{ f(x) : f(x) = e^{sx} , \quad s \in \mathbb{R} \right\}$$

I) Identity $f(x) = e^{0x} = 1 \in \tilde{\mathcal{F}}$ ✓

II) Separability $f(x) = e^{1x}$ $f(x_1) \neq f(x_2)$ if $x_1 \neq x_2$ ✓

III) Algebraic Closure $f(x) = e^{sx} \quad g(x) = e^{tx}$

$$a f(x) + b g(x) = a e^{sx} + b e^{tx} \neq e^{vx} \text{ for any } v \in$$

So additive closure <u>fails</u>

$$f(x) g(x) = e^{sx} e^{tx} = e^{(s+t)x} \in \tilde{\mathcal{F}} \checkmark$$