Approximation Theory—

23 May 1990 Stone - Weierstrass Theorem examples— Partial fractions

*Neil E Cotter*

ex: Partial fractions in one variable satisfy S-W thm:

$$\mathcal{F} = \left\{ f(x) = \sum_{n=1}^{\infty} \frac{b_n}{1+a_n x} \right\} \qquad \begin{matrix} a_n \in \mathbb{R}, & b_n \in \mathbb{R}, & x \in [0,1] \\ a_n > 0 & b_n > 0 & \end{matrix}$$

I) Identity $\quad a_n = 0 \quad b_n = 1$ gives $f(x) = 1 \in \mathcal{F}$ ✓

II) Separability $\quad f(x) = \dfrac{1}{1+x}$ is monotonically decreasing $\in \mathcal{F}$ ✓

III) Closure

$$f(x) = \sum_{n=1}^{\infty} \frac{b_n}{1+a_n x} \qquad g(x) = \sum_{m=1}^{\infty} \frac{d_m}{1+c_m x}$$

Additive

$$a f(x) + b g(x) = \sum_{n=1}^{\infty} \left( \frac{a b_n}{1+a_n x} + \frac{b d_n}{1+c_n x} \right)$$

Let

$$h_k = \begin{cases} a b_{k/2} & \text{for } k \text{ even} \\ b d_{(k+1)/2} & \text{" " odd} \end{cases}$$

(Renumber terms in $\Sigma$. Take $ab_n$ term then $bd_n$ term, alternat)

$$e_k = \begin{cases} a_{k/2} & k \text{ even} \\ c_{(k+1)/2} & k \text{ odd} \end{cases}$$

$$a f(x) + b g(x) = \sum_{k=1}^{\infty} \frac{h_k}{1+e_k x} \in \mathcal{F} \quad ✓$$

Multiplicative

$$f(x)\, g(x) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{b_n}{1+a_n x} \frac{d_m}{1+c_m x}$$

$$\boxed{\text{recall} \quad \frac{1}{1+ax} \frac{1}{1+bx} = \frac{a/(a-b)}{1+ax} + \frac{b/(b-a)}{1+bx}}$$

$$= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{b_n d_m a_n/(a_n - c_m)}{1+a_n x} + \frac{b_n d_m c_m/(c_m - a_n)}{1+c_m x}$$

Can write $\Sigma\Sigma$ as $\Sigma$

and can turn $\Sigma\,(\text{term}_1 + \text{term}_2)$

into $\Sigma\ \text{term}_k$ as above.

Conclusion: $f(x)\, g(x) \in \mathcal{F}$ ✓