Coefficients from inner products

thm: If $\mathcal{G}$ is an orthonormal basis for space $\mathcal{F}$, then for any $f \in \mathcal{F}$ we have

($\mathcal{G} = \{g_n : n = 1, \dots, \infty\}$)

$$f = \sum_{n=1}^{\infty} a_n g_n$$

where the coefficients are given by inner products as follows:

$$a_n = (f, g_n)$$

pf: Since $\mathcal{G}$ is a basis, there exists a set of coefficients for the series expansion of f:

$$f = \sum_{n=1}^{\infty} a_n g_n$$

~~[illegible]~~

But $(f, g_n) = \sum_{m=1}^{\infty} (a_m g_m, g_n) = \sum_{m=1}^{\infty} a_m (g_m, g_n) = a_n (g_n, g_n) = a_n \cdot 1 = a_n$

We have found the unique set of a_n's. all other g_m, g_n are $\perp$

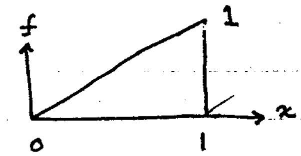
– Coefficients from inner products (cont.)

ex: Fourier series, $\mathcal{B} = \{ \overset{2}{\cancel{\sqrt{2}}} \cos 2\pi n x, \overset{2}{\cancel{\sqrt{2}}} \sin 2\pi n x : n = 0, \dots, \infty \}$
is an orthonormal basis for
$\mathcal{F}$ = continuous real-valued periodic functions on $[0,1]$

Then for $f \in \mathcal{F}$ we have $f = \sum_{n=0}^{\infty} a_n \overset{2}{\cancel{\sqrt{2}}} \cos 2\pi n x + b_n \overset{2}{\cancel{\sqrt{2}}} \sin 2\pi n x$

where $a_n = (f, \overset{2}{\cancel{\sqrt{2}}} \cos 2\pi n x)$ $\quad b_n = (f, \overset{2}{\cancel{\sqrt{2}}} \sin 2\pi n x)$

ex: find: Fourier coefficients for f:
$f(x) = x$



soln: $a_n = (f, \overset{2}{\cancel{\sqrt{2}}} \cos 2\pi n x) = \int_0^1 x \cdot \overset{2}{\cancel{\sqrt{2}}} \cos 2\pi n x \, dx$

$\int x \cos x = \cos x + x \sin x$

$$a_n = \frac{\sqrt{2}}{(2\pi n)^2} \int_0^1 2\pi n x \cos 2\pi n x \, d\, 2\pi n x$$

$$= \frac{\sqrt{2}}{(2\pi n)^2} \int_0^{2\pi n} y \cos y \, dy$$

$$= \frac{\sqrt{2}}{(2\pi n)^2} (\cos y + y \sin y) \Big|_0^{2\pi n}$$

$$= \frac{\sqrt{2}}{(2\pi n)^2} [(1+0) - (1+0)]$$

$$= 0$$

$$b_n = (f, \sqrt{2} \sin 2\pi n x) = \int_0^1 x \sqrt{2} \sin 2\pi n x \, dx$$

$\int x \sin x = \sin x - x \cos x$

$$b_n = \frac{\sqrt{2}}{(2\pi n)^2} (\sin y - y \cos y) \Big|_0^{2\pi n} = \frac{\sqrt{2}}{(2\pi n)^2} ((0 - 2\pi n) - (0 - 0))$$

$$= \frac{-\sqrt{2}}{(2\pi n)^2} \cdot 2\pi n = \frac{-\sqrt{2}}{2\pi n}$$

Function Spaces

— Coefficients from inner products (cont.)

ex: Orthonormal Basis vectors for $\mathbb{R}^3$, $\mathcal{J} = \{[1,0,0], [0,1,0], [0,0,1]\}$

Then for $\vec{v} \in \mathbb{R}$ we have
$$\vec{v} = (\vec{v}, [1,0,0]) \cdot [1,0,0] + (\vec{v}, [0,1,0]) \cdot [0,1,0] + (\vec{v}, [0,0,1]) \cdot [0,0,1]$$

ex: $\vec{v} = [2,3,4]$
$$(\vec{v}, [1,0,0]) = [2,3,4] \circ [1,0,0] = 2 \cdot 1 + 3 \cdot 0 + 4 \cdot 0 = 2$$
$$(\vec{v}, [0,1,0]) = [2,3,4] \circ [0,1,0] = 2 \cdot 0 + 3 \cdot 1 + 4 \cdot 0 = 3$$
$$(\vec{v}, [0,0,1]) = [2,3,4] \circ [0,0,1] = 2 \cdot 0 + 3 \cdot 0 + 4 \cdot 1 = 4$$

$$\therefore \vec{v} = 2 \cdot [1,0,0] + 3[0,1,0] + 4[0,0,1]$$

ex: Orthonormal basis vectors for $\mathbb{R}^3$, $\mathcal{G} = \left\{ \underset{g_1}{\begin{bmatrix} \cos\theta \cos\varphi \\ \sin\theta \cos\varphi \\ \sin\varphi \end{bmatrix}}, \underset{g_2}{\begin{bmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{bmatrix}}, \underset{g_3}{\begin{bmatrix} -\cos\theta \sin\varphi \\ -\sin\theta \sin\varphi \\ \cos\varphi \end{bmatrix}} \right\}$

(Polar coordinate rotations)

Check for orthonormal:

$$\|g_1\|^2 = g_1 \circ g_1 = \cos^2\theta \cos^2\varphi + \sin^2\theta \cos^2\varphi + \sin^2\varphi = \cos^2\varphi + \sin^2\varphi = 1 \checkmark \quad (\sin^2 + \cos^2 = 1)$$

$$(g_1, g_2) = g_1 \circ g_2 = -\sin\theta \cancel{\cos\theta} \cos\varphi + \cos\theta \cancel{\sin\theta} \cos\varphi + 0 = 0 \checkmark$$

$$(g_1, g_3) = g_1 \circ g_3 = -\cos^2\theta \cos\varphi \sin\varphi - \sin^2\theta \cos\varphi \sin\varphi + \sin\varphi \cos\varphi = -1 \cdot \cos\varphi \sin\varphi + \sin\varphi \cos\varphi = 0 \checkmark$$

$$\|g_2\|^2 = g_2 \circ g_2 = \sin^2\theta + \cos^2\theta = 1 \checkmark$$

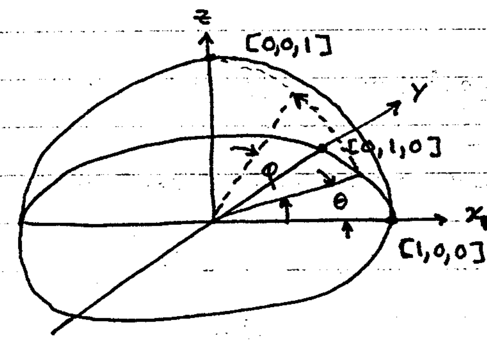
$$(g_2, g_3) = g_2 \circ g_3 = \sin\theta \cos\theta \sin\varphi - \cos\theta \sin\theta \sin\varphi = 0 \checkmark$$

$$\|g_3\|^2 = g_3 \circ g_3 = \cos^2\theta \sin^2\varphi + \sin^2\theta \sin^2\varphi + \cos^2\varphi = 1 \checkmark$$

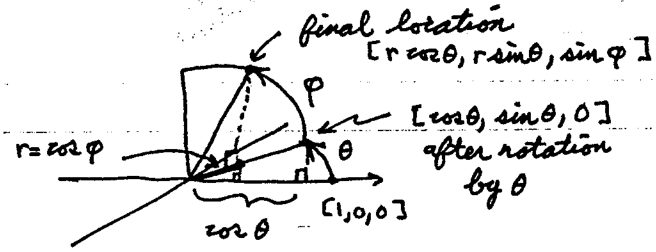
(ex: cont.)

picture of polar rotations:

Rotate by θ in x-y plane
Then rotate toward north-pole by φ.

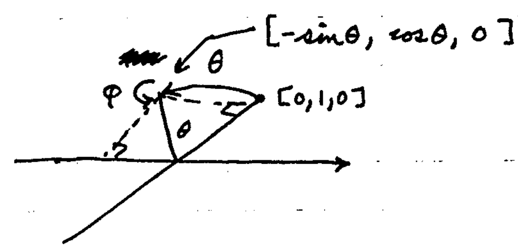


Observe what happens to [1,0,0] vector:



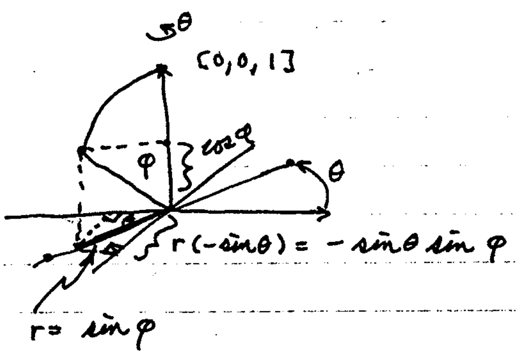
It becomes $[r\cos\theta, r\sin\theta, \sin\varphi]$
$= [\cos\varphi\cos\theta, \cos\varphi\sin\theta, \sin\varphi] = g_1$

Observe what happens to [0,1,0] vector:



It becomes $[-\sin\theta, \cos\theta, 0] = g_2$ after θ rotation, and it acts as axel for φ rotation so it doesn't change further.

Observe what happens to [0,0,1] vector:



It acts as the axel for the θ rotation. Then it rotates down by φ along the line defined by the rotated [1,0,0] vector.
So we get $[-\cos\theta\sin\varphi, -\sin\theta\sin\varphi, \cos\varphi] = g_3$

We have just rotated the basis vectors.
But they are still $\perp$ and have unit length.

Function Spaces

– Coefficients from inner products (cont.)

(ex cont.)

$$\vec{v} = [2,3,4] = (\vec{v}, g_1)\cdot g_1 + (\vec{v}, g_2)\cdot g_2 + (\vec{v}, g_3)\cdot g_3$$

$$(\vec{v}, g_1) = [2,3,4]\circ g_1 = 2\cos\theta\cos\varphi + 3\sin\theta\cos\varphi + 4\sin\varphi \equiv a_1$$

$$(\vec{v}, g_2) = [2,3,4]\circ g_2 = -2\sin\theta + 3\cos\theta \equiv a_2$$

$$(\vec{v}, g_3) = [2,3,4]\circ g_3 = -2\cos\theta\sin\varphi - 3\sin\theta\sin\varphi + 4\cos\varphi \equiv a_3$$

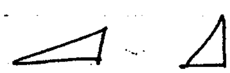
$$\vec{v} = a_1\cdot g_1 + a_2\cdot g_2 + a_3\cdot g_2$$

check $$\vec{v} = \begin{bmatrix} a_1\cos\theta\cos\varphi \\ a_1\sin\theta\cos\varphi \\ a_1\sin\varphi \end{bmatrix} + \begin{bmatrix} -a_2\sin\theta \\ a_2\cos\theta \\ 0 \end{bmatrix} + \begin{bmatrix} -a_3\cos\theta\sin\varphi \\ -a_3\sin\theta\sin\varphi \\ a_3\cos\varphi \end{bmatrix}$$

$$\vec{v} = \begin{bmatrix} a_1\cos\theta\cos\varphi - a_2\sin\theta - a_3\cos\theta\sin\varphi \\ a_1\sin\theta\cos\varphi + a_2\cos\theta - a_3\sin\theta\sin\varphi \\ a_1\sin\varphi + a_3\cos\varphi \end{bmatrix} \text{ should } = [2,3,4]$$

We can leave things in this form for general θ, φ, or if we choose particular θ and φ values we can get actual numbers.

Say $\theta = 30°$ $\varphi = 45°$



$\sin\theta = \frac{1}{2}$ $\sin\varphi = 1/\sqrt{2}$

$\cos\theta = \frac{\sqrt{3}}{2}$ $\cos\varphi = 1/\sqrt{2}$

$$a_1 = \cancel{2}\cdot\frac{\sqrt{3}}{\cancel{2}}\frac{1}{\sqrt{2}} + 3\frac{1}{2}\frac{1}{\sqrt{2}} + \frac{4}{\sqrt{2}} = \frac{1}{\sqrt{2}}\left(\sqrt{3} + \frac{3}{2} + 4\right)$$

$$a_2 = -2\frac{1}{2} + 3\frac{\sqrt{3}}{2} = \frac{1}{2}\left(3\sqrt{3} - 2\right)$$

$$a_3 = -\cancel{2}\frac{\sqrt{3}}{\cancel{2}}\frac{1}{\sqrt{2}} - 3\frac{1}{2}\frac{1}{\sqrt{2}} + \frac{4}{\sqrt{2}} = \frac{1}{\sqrt{2}}\left(-\sqrt{3} - \frac{3}{2} + 4\right)$$

$\vec{v} = a_1 g_1 + a_2 g_2 + a_3 g_3$ where we have found the coefficients a_1, a_2, a_3.