Function Spaces

- Orthogonality

notn: $f_1 \perp f_2 \equiv f_1$ is orthogonal to f_2

def: $f_1 \perp f_2 \equiv (f_1, f_2) = 0 \equiv$ inner product of $f_1, f_2 = 0$

ex: $\sin 2\pi x \perp \cos 2\pi x$ on interval $[0,1]$

$$(\sin 2\pi x, \cos 2\pi x) = \int_0^1 \sin 2\pi x \cdot \cos 2\pi x \, dx$$

Use $\sin A \cos B = \frac{1}{2} \sin(A+B) + \frac{1}{2} \sin(A-B)$

$$\sin 2\pi x \cos 2\pi x = \frac{1}{2} \sin 4\pi x$$

$$\therefore (\sin 2\pi x, \cos 2\pi x) = \int_0^1 \frac{1}{2} \sin 4\pi x \, dx$$

$$= -\frac{1}{2} \frac{1}{4\pi} \cos 4\pi x \Big|_0^1$$

$$= -\frac{1}{2} \frac{1}{4\pi} (1 - 1)$$

$$= 0 \checkmark$$

ex: $\begin{bmatrix} 1 \\ -1 \end{bmatrix} \perp \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ for vector space $\mathbb{R}^2$

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix} \circ \begin{bmatrix} 2 \\ 2 \end{bmatrix} = 1 \cdot 2 + -1 \cdot 2 = 2 - 2 = 0 \checkmark$$

ex: $\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} \perp \begin{bmatrix} \sin\theta \\ -\cos\theta \end{bmatrix}$

$$\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} \circ \begin{bmatrix} \sin\theta \\ -\cos\theta \end{bmatrix} = \cos\theta \sin\theta - \sin\theta \cos\theta = 0 \checkmark$$