Find $\mathcal{L}\{g(t)\}$ for a) $g(t) = \dfrac{d}{dt} \sin \omega t$

b) $g(t) = \dfrac{d}{dt} \cos \omega t$

c) $g(t) = \dfrac{d^3}{dt^3} t^2$ (and $\dfrac{d^3}{dt^3}[t^2 u(t)]$)

d) Check answers by taking derivatives first and then computing $\mathcal{L}\{\ \}$.

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ans: a) $\dfrac{s\omega}{s^2+\omega^2}$ b) $\dfrac{s^2}{s^2+\omega^2} - 1$ c) 0 for $\dfrac{d^3}{dt^3} t^2$ and 2 for $\dfrac{d^3}{dt^3}[t^2 u(t)]$

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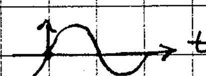
sol'n: a) $\mathcal{L}\left\{\dfrac{d}{dt} f(t)\right\} = sF(s) - f(0^-)$ where $\mathcal{L}\{f(t)\} = F(s)$

Here, $f(t) = \sin \omega t$ $F(s) \equiv \mathcal{L}\{\sin \omega t\} = \dfrac{\omega}{s^2+\omega^2}$ (from table)

$$\therefore \mathcal{L}\left\{\frac{d}{dt} \sin \omega t\right\} = \frac{s\omega}{s^2+\omega^2} - \sin(\underbrace{\omega \cdot 0^-}_{0})$$

$\sin(0) = 0$



$$= \frac{s\omega}{s^2+\omega^2}.$$

b) $f(t) = \cos \omega t$ $\mathcal{L}\{\cos \omega t\} = \dfrac{s}{s^2+\omega^2} \equiv F(s)$

$$\therefore \mathcal{L}\left\{\frac{d}{dt} \cos \omega t\right\} = \frac{s^2}{s^2+\omega^2} - \underbrace{\cos \omega \cdot 0}_{1} = \frac{s^2}{s^2+\omega^2} - 1 = \frac{-\omega^2}{s^2+\omega^2}$$

Note: Solution manual gives answer as just $\dfrac{s^2}{s^2+\omega^2}$

This is the answer for $f(t) = \cos(\omega t) \cdot u(t)$:

$$\mathcal{L}\left\{\frac{d}{dt}[\cos(\omega t) \cdot u(t)]\right\} = s \cdot F(s) - \overbrace{\underbrace{\cos(\omega \cdot 0^-)}_{1} \cdot \underbrace{u(0^-)}_{0}}^{0}$$

New term

$$\text{where } F(s) = \mathcal{L}\{\cos(\omega t)\, u(t)\} = \int_0^\infty \cos(\omega t)\, u(t)\, e^{-st}\, dt$$

$$= \int_{0^-}^\infty \cos(\omega t)\, e^{-st}\, dt = \mathcal{L}\{\cos(\omega t)\} = \frac{s}{s^2+\omega^2}.$$

$$\therefore \mathcal{L}\left\{\frac{d}{dt}[\cos(\omega t)\cdot u(t)]\right\} = sF(s) = \frac{s^2}{s^2+\omega^2}$$

Comment: The Text book is careless about specifying whether we have $\mathcal{L}\{f(t)\}$ or $\mathcal{L}\{f(t)\cdot u(t)\}$. These are very different when we start taking derivatives. Thus, we will assume $f(t)$ means $f(t)$. If we mean $f(t)u(t)$, we will say $f(t)u(t)$.

c) $$\mathcal{L}\left\{\frac{d^3}{dt^3}t^2\right\} = s^3F(s) - s^2f(0^-) - s^1\frac{d}{dt}f(t)\Big|_{t=0^-} - s^0\frac{d^{n-1=2}}{dt^{n-1=2}}f(t)\Big|_{t=0^-}$$

where $F(s) \equiv \mathcal{L}\{t^2\} = \frac{2}{s^3}$.

Note: $\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$ from $\mathcal{L}\{t\cdot t^{n-1}\} = -\frac{d\,F(s)}{ds}$

applied repeatedly starting with $\mathcal{L}\{t\} = \frac{1}{s^2}$ $\left(= -\frac{d}{ds}\mathcal{L}\{u(t)\}\right.$

$$= -\frac{d}{ds}\frac{1}{s} = -\frac{d}{ds}s^{-1} = -(-1)s^{-2}$$

$$\left.= \frac{1}{s^2}\right) \checkmark$$

Comment: $t = \int_0^t u(t)\,dt$ and it follows that

$$\mathcal{L}\{t\} = \frac{\mathcal{L}\{u(t)\}}{s} = \frac{1/s}{s} = \frac{1}{s^2}$$

(from the operational transform $\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\} = \frac{F(s)}{s}$)

$\therefore$ When we integrate, we multiply the Laplace transform by $\frac{1}{s}$. (When we differentiate, we multiply by s and account for steps by adding $-f(0^-)$.)

Here, $f(0^-) = t^2\big|_{0^-} = 0$ $\quad \frac{d}{dt}f(t)\Big|_{t=0^-} = 2t\Big|_{t=0^-} = 0$

$$\frac{d^2}{dt^2}f(t)\Big|_{t=0^-} = 2\Big|_{t=0^-} = 2 \qquad \underbrace{\frac{d^3}{dt^3} = 0\Big|_{t=0^-} = 0}_{\text{not needed here, see (d)}}$$

$$\therefore \mathcal{L}\left\{\frac{d^3}{dt^3}t^2\right\} = s^3\cdot\frac{2}{s^3} - \underbrace{s^2\cdot 0}_{0} - \underbrace{s^1\cdot 0}_{0} - \underbrace{s^0}_{=1}\cdot 2$$

$$= 2 - 2 = 0$$

Now consider $\mathcal{L}\left\{\frac{d^3}{dt^3}[t^2\cdot u(t)]\right\}$

$$= s^3F(s) - s^2 f(0^-) - s^1\frac{d}{dt}f(t)\Big|_{t=0^-} - s^0\frac{d^2}{dt^2}f(t)\Big|_{t=0^-}$$

where $F(s) \equiv \mathcal{L}\{t^2\cdot u(t)\} = \mathcal{L}\{t^2\} = \frac{2!}{s^3} = \frac{2}{s^3}$

and $f(t) = t^2\cdot u(t)$ $\quad f(0^-) = 0\cdot 0 = 0$

Now for derivatives of $f(t)$:

$$\frac{d}{dt}t^2u(t) = 2t\,u(t) + t^2\delta(t) \qquad = 0 \text{ at } t = 0^-$$

$$\frac{d^2}{dt^2}t^2u(t) = \frac{d}{dt}\text{ " } = 2u(t) + \overbrace{2t\,\delta(t) + 2t\,\delta(t)}^{4t\,\delta(t)} + t^2\delta'(t)$$

$$= 2\cdot 0 + 2\cdot 0\cdot 0 + 2\cdot 0\cdot 0 + 0^2\cdot 0 = 0 \text{ at } t = 0^-$$

$$\underbrace{\frac{d^3}{dt^3}t^2u(t) = \frac{d}{dt}}_{\text{not needed here, see (d)}}\text{ " } = 2\delta(t) + 4\delta(t) + 4t\,\delta'(t) + 2t\,\delta'(t) + t^2\delta''(t)$$

$$= 6\delta(t) + 6t\delta'(t) + t^2\delta''(t)$$

$$= 6\cdot 0 + 6\cdot 0\cdot 0 + 0^2\cdot 0 \text{ at } t = 0^-$$

$$\therefore \mathcal{L}\left\{\frac{d^3}{dt^3}[t^2u(t)]\right\} = s^3\cdot\frac{2}{s^3} - s^2\cdot 0 - s^1\cdot 0 - \underbrace{s^0}_{=1}\cdot 0$$

$$= 2 \qquad \text{different answer!}$$

d) Take derivatives first, then compute $\mathcal{L}\{\ \}$ to get verification of above results.

$$\mathcal{L}\left\{\frac{d}{dt}\sin\omega t\right\} = \mathcal{L}\{\omega\cos\omega t\} = \frac{\omega s}{s^2+\omega^2} \quad \checkmark \quad (a)$$

$$\mathcal{L}\left\{\frac{d}{dt}\cos\omega t\right\} = \mathcal{L}\{-\omega\sin\omega t\} = \frac{-\omega\cdot\omega}{s^2+\omega^2} \quad \checkmark \quad (b)$$

$$\mathcal{L}\left\{\frac{d}{dt}[\cos(\omega t)\,u(t)]\right\} = \mathcal{L}\{-\omega\sin(\omega t)\,u(t) + \cos(\omega t)\,\delta(t)\}$$

product rule for $\frac{d}{dt}$, and $\frac{d}{dt}u(t) = \delta(t)$

$= \mathcal{L}\{-\omega\sin(\omega t)\}$ ($u(t)$ doesn't affect $\mathcal{L}\{\}$ unless we start differentiating it)

$+\mathcal{L}\{\cos(\omega t)\,\delta(t)\}$

$$= \frac{-\omega\cdot\omega}{s^2+\omega^2} + \int_{0^-}^{\infty}\cos(\omega t)\,\delta(t)\,e^{-st}\,dt$$

Note: $\int_{0^-}^{\infty} f(t)\,\delta(t)\,dt = f(0)$

The delta function picks out value of $f(t)$ at $t=0$.

Take product of pulse and $f(t)$. Then integrate. Let pulse width $\to 0$, height $\to\infty$, area $= 1$ always for pulse.

Answer in $\lim_{\epsilon\to 0}\int_{0^-}^{\infty} f(t)\cdot \text{pulse}(t)\,dt$

$$\equiv \int_{0^-}^{\infty} f(t)\,\delta(t)\,dt = f(0)$$

$$= \frac{-\omega^2}{s^2+\omega^2} + \int_{0^-}^{\infty}\underbrace{\cos(\omega t)e^{-st}}_{f(t)}\,\delta(t)\,dt$$

$$= \frac{-\omega^2}{s^2+\omega^2} + \underset{1}{\cos(\omega\cdot 0)}\cdot\underset{1}{e^{-s\cdot 0}}$$

$$= \frac{-\omega^2}{s^2+\omega^2} + 1 = \frac{-\omega^2}{s^2+\omega^2} + \frac{s^2+\omega^2}{s^2+\omega^2} = \frac{s^2}{s^2+\omega^2} \quad \checkmark$$

$$\mathcal{L}\left\{\frac{d^3}{dt^3}t^2\right\} = \mathcal{L}\left\{\frac{d^2}{dt^2}2t\right\} = \mathcal{L}\left\{\frac{d}{dt}2\right\} = \mathcal{L}\{0\} = 0 \quad \checkmark$$

$$\mathcal{L}\left\{\frac{d^3}{dt^3}\left[t^2u(t)\right]\right\} = \mathcal{L}\{6\delta(t) + 6t\underbrace{\delta'(t)}_{\frac{d}{dt}\delta(t)} + t^2\underbrace{\delta''(t)}_{\frac{d^2}{dt^2}\delta(t)}\}$$

derived in part (c)

One approach: • $\mathcal{L}\{6\delta(t)\} = 6\mathcal{L}\{\delta(t)\} = 6$

• $\mathcal{L}\{6t\delta'(t)\} = 6\mathcal{L}\{t f(t)\} = 6\left[\frac{-dF(s)}{ds}\right]$

where $F(s) = \mathcal{L}\{\delta'(t)\} = s\cdot\underbrace{\mathcal{L}\{\delta(t)\}}_{1} - \delta(t=0^-)$

$\delta(t=0^-) = 0$ since $\delta(t) \uparrow \longrightarrow t$

$\delta'(t)$ is 0 except at 0.

$\therefore \quad \mathcal{L}\{\delta'(t)\} = s \qquad \mathcal{L}\{6t\delta'(t)\} = 6\left[\frac{-d}{ds}s\right] = -6$

• $\mathcal{L}\{t^2\delta''(t)\} = (-1)^2\frac{d^2}{ds^2}\mathcal{L}\{\delta''(t)\}$

We have $\mathcal{L}\{\delta''(t)\} = s^2\underbrace{\mathcal{L}\{\delta(t)\}}_{1} - s^1\underbrace{\delta(0^-)}_{0} - s^0\underbrace{\frac{d}{dt}\delta(t)\Big|_{t=0^-}}_{0}$

Note: $\frac{d\delta(t)}{dt} = 0$ unless $t=0$

$\therefore \frac{d}{dt}\delta(t)\Big|_{t=0^-} = 0$

$\therefore \mathcal{L}\{\delta''(t)\} = s^2$

$\mathcal{L}\{t^2\delta''(t)\} = (-1)^2\frac{d^2}{ds^2}s^2 = 1\cdot\frac{d}{ds}2s = 1\cdot 2 = 2$

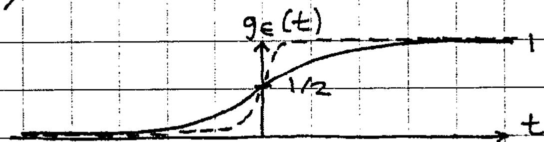
Adding the results, we have

$$\mathcal{L}\left\{\frac{d^3}{dt^3}\left[t^2u(t)\right]\right\} = 6 - 6 + 2 = 2 \quad \checkmark$$

Another approach: In the above approach we still had to use the operational transform for $\frac{d}{dt}$.

Unfortunately, we are using the tool whose validity we are supposed to verify!

To avoid this problem, we need an infinitely differentiable function that becomes the step function in the limit. Call this function $g_\epsilon(t)$ for a given ϵ. It must be smooth no matter how many times we take its derivative:



The reason we need such smoothness is that $\frac{d^n}{dt^n}[g_\epsilon(t) \cdot f(t)]$ will be smooth if $f(t)$ is smooth.

In other words, we will not be creating steps or impulses when we take higher order derivatives of this new type of step function multiplied by a smooth $f(t)$.

Then we may compute $\mathcal{L}\{f(t)u(t)\}$ as $\lim_{\epsilon \to 0} \int_{0^-}^{\infty} f(t) g_\epsilon(t) e^{-st} dt$,

and we will not be integrating impulses.

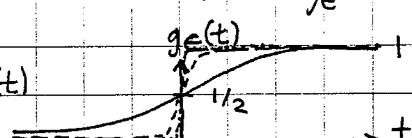
We use $g_\epsilon(t) = \dfrac{1}{1 + e^{-t/\epsilon}}$.

As $t \to -\infty$ we have $g_\epsilon(t) \to \dfrac{1}{1 + e^{+\infty}} = 0$ ✓

As $t \to \infty$ we have $g_\epsilon(t) \to \dfrac{1}{1 + e^{-\infty}} = \dfrac{1}{1}$ ✓

For $t = 0$ we have $g_\epsilon(t) = \dfrac{1}{1 + e^{0}} = \dfrac{1}{2}$ ✓

$g_\epsilon(t) \to u(t)$
as $\epsilon \to 0$



✓

If we consider $\mathcal{L}\left\{\frac{d^3}{dt^3}\left[t^2 \cdot g_\epsilon(t)\right]\right\}$

We find that $\frac{d^3}{dt^3}\left[t^2 \cdot g_\epsilon(t)\right] = 6g_\epsilon'(t) + 6tg_\epsilon''(t) + t^2 g_\epsilon'''(t)$

(Same derivative as before, with $g_\epsilon'(t)$ replacing $\delta(t)$.

- $\mathcal{L}\{6g_\epsilon'(t)\} = 6\int_{0^-}^{\infty} g_\epsilon'(t)\,dt$ in $\lim_{\epsilon \to 0}$

$\lim_{\epsilon \to 0} \int_{0^-}^{\infty} g_\epsilon'(t)\,dt = 1$ since $g_\epsilon'(t)$ is the derivative of a function that approaches a step function.

- $\mathcal{L}\{6tg_\epsilon''(t)\} = 6\int_{0^-}^{\infty} tg_\epsilon''(t)\,dt$ in $\lim_{\epsilon \to 0}$

Use $\int$ by parts: $\int_a^b uv' = u \cdot v\Big|_a^b - \int_a^b u'v.$

here $u = t \quad v' = g_\epsilon''(t) \Rightarrow v = g_\epsilon'(t)$

$$\therefore \lim_{\epsilon \to 0} \int_{0^-}^{\infty} tg_\epsilon''(t) = t \cdot g_\epsilon'(t)\Big|_{0^-}^{\infty} - \underbrace{\int_{0^-}^{\infty} 1 \cdot g_\epsilon'(t)\,dt}_{1 \text{ from above results}}$$

Now consider $t \cdot g_\epsilon'(t)$.

$$g_\epsilon'(t) = \frac{d}{dt}\frac{1}{1+e^{-t/\epsilon}} = \frac{d}{dt}\left[1+e^{-t/\epsilon}\right]^{-1}$$

$$= -1\left[1+e^{-t/\epsilon}\right]^{-2} \cdot \underbrace{\frac{d}{dt}\left[1+e^{-t/\epsilon}\right]}_{-\frac{1}{\epsilon}e^{-t/\epsilon}}$$

$$= \frac{1}{\epsilon}\frac{1}{1+e^{-t/\epsilon}} \cdot \frac{e^{-t/\epsilon}}{1+e^{-t/\epsilon}} = \frac{1}{\epsilon}g_\epsilon(t)\left[1-g_\epsilon(t)\right]$$

Now, as $t \to \infty$ the denominator of $g_\epsilon'(t) \to 1$.

$$\therefore \lim_{t \to \infty} tg_\epsilon'(t) = \lim_{t \to \infty} t \cdot e^{-t/\epsilon}$$

By L'hopital's rule $\lim_{t\to\infty} t\cdot e^{-t/\epsilon} = \lim_{t\to\infty} \dfrac{\frac{dt}{dt}}{\frac{d}{dt}e^{t/\epsilon}} = \lim_{t\to\infty} \dfrac{1}{\frac{1}{\epsilon}e^{t/\epsilon}}$

$$= \frac{1}{\infty} = 0.$$

At $t=0^-$ we have $t\cdot g_\epsilon'(t) = 0^-\cdot \frac{1}{\epsilon} g_\epsilon(0^-)[1-g_\epsilon(0^-)]$

$$= 0^-\cdot \frac{1}{\epsilon}\,\frac{1}{1+e^{-0^-/\epsilon}}\left(1-\frac{1}{1+e^{-0^-/\epsilon}}\right)$$

$$= 0^-\cdot\frac{1}{\epsilon}\cdot\frac{1}{2}\cdot\frac{1}{2}$$

$$= 0^- = 0$$

Thus $\int_{0^-}^{\infty} t g_\epsilon''(t)\,dt = 0 - 0 - 1 = -1.$

- $\mathcal{L}\{t^2 g_\epsilon'''(t)\} = \int_{0^-}^{\infty} t^2 g_\epsilon'''(t)\,dt$ in $\lim_{\epsilon\to 0}$

Use $\int$ by parts, $u = t^2$ $v' = g_\epsilon''' \Rightarrow v = g_\epsilon''$:

$$\lim_{\epsilon\to 0}\left[t^2 g_\epsilon''(t)\Big|_0^\infty - \underbrace{\int_{0^-}^{\infty} 2t\, g_\epsilon''(t)\,dt}_{-2 \text{ from above results}}\right]$$

Using $g_\epsilon''(t) = \frac{1}{\epsilon^2} g_\epsilon(t)[1-g_\epsilon(t)][1-2g_\epsilon(t)]$

we can show that $\lim_{t\to\infty} t^2 g_\epsilon''(t) = 0$

and $t^2 g_\epsilon''(t) = 0$ at $t = 0^-$.

Thus $\lim_{\epsilon\to 0}\int_0^\infty t^2 g_\epsilon'''(t)\,dt = -2$

Adding our results gives $\mathcal{L}\left\{\frac{d^3}{dt^3} t^2 g_\epsilon(t)\right\} = 6 - 6 - (-2) = 2.$ ✓