EX: Find the Laplace transform of the following waveform:

$$f(t) = t\frac{d}{dt}\left(te^{-at}\right)$$

sol'n: Although we might take the derivative before taking the Laplace transform, it is generally simpler to use Laplace transform identities.

We work from the inside out.

From a table of basic transforms, we have

$$\mathcal{L}\{te^{-at}\} = \frac{1}{(s+a)^2}$$

We now apply the identity for derivatives:

$$\mathcal{L}\left\{\frac{d}{dt}f(t)\right\} = s\mathcal{L}\{f(t)\} - f(0^-)$$

Thus, $$\mathcal{L}\left\{\frac{d}{dt}\left(te^{-at}\right)\right\} = \frac{s}{(s+a)^2} - \underbrace{\left.\left(te^{-at}\right)\right|_{t=0^-}}_{=0}$$

We now apply the identity for multiplication by t:

$$\mathcal{L}\{tf(t)\} = -\frac{d}{ds}\mathcal{L}\{f(t)\}$$

Thus, $$\mathcal{L}\left\{t\frac{d}{dt}\left(te^{-at}\right)\right\} = \frac{-d}{ds}\frac{s}{(s+a)^2}$$

or $\mathcal{L}\left\{t\frac{d}{dt}\left(te^{-at}\right)\right\} = -\frac{d}{ds}\left[s\cdot(s+a)^{-2}\right]$

$$= -\left[(s+a)^{-2} + s(-2)(s+a)^{-3}\right]$$

$$= -\left[\frac{s+a}{(s+a)^3} - \frac{2s}{(s+a)^3}\right]$$

$$\mathcal{L}\left\{t\frac{d}{dt}\left(te^{-at}\right)\right\} = \frac{s-a}{(s+a)^3}$$