
EX: Find the Laplace transform of the following waveform:

$$f(t) = e^{-at}\int_0^t t\sin(\omega t)dt$$

soln: We work from the inside out, starting with the Laplace transform of $\sin(\omega t)$ from a table of transform pairs:

$$\mathcal{L}\{\sin(\omega t)\} = \frac{\omega}{s^2+\omega^2}$$

We now apply the identity for multiplication by t:

$$\mathcal{L}\{tf(t)\} = -\frac{d}{ds}\mathcal{L}\{f(t)\}$$

$$\mathcal{L}\{t\sin(\omega t)\} = -\frac{d}{ds}\frac{\omega}{s^2+\omega^2}$$

$$= -\frac{d}{ds}\left(\omega\left[s^2+\omega^2\right]^{-1}\right)$$

$$= -\omega(-1)\left[s^2+\omega^2\right]^{-2}2s$$

$$\mathcal{L}\{t\sin(\omega t)\} = \frac{2\omega s}{\left[s^2+\omega^2\right]^2}$$

We now apply the identity for $\int_0^t$:

$$\mathcal{L}\left\{\int_0^t f(t)dt\right\} = \frac{1}{s}\mathcal{L}\{f(t)\}$$

$$\mathcal{L}\left\{\int_0^t t\sin(\omega t)\,dt\right\} = \frac{2\omega s}{s\left[s^2+\omega^2\right]^2}$$

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We now apply the identity for multiplication by e^{-at}:

$$\mathcal{L}\{e^{-at} f(t)\} = \mathcal{L}\{f(t)\}\Big|_{s+a \text{ replaces } s}$$

$$\mathcal{L}\left\{e^{-at}\int_0^t t \sin(\omega t)\, dt\right\} = \frac{2\omega}{\left[(s+a)^2 + \omega^2\right]^2}$$