Ref: Nilsson & Riedel, Rev 6, Prob 12.26

Find $f(t) = \mathcal{L}^{-1}\{F(s)\}$ for a) $F(s) = \dfrac{18s^2 + 66s + 54}{(s+1)(s+2)(s+3)}$

b) $F(s) = \dfrac{25s^2 + 86s + 40}{s(s+2)(s+4)}$

c) $F(s) = \dfrac{11s^2 + 172s + 700}{(s+2)(s^2+12s+100)}$

d) $F(s) = \dfrac{56s^2 + 112s + 5000}{s(s^2 + 14s + 625)}$

ans: a) $f(t) = \left[3e^{-t} + 6e^{-2t} + 9e^{-3t}\right]u(t)$

b) $f(t) = \left[5 + 8e^{-2t} + 12e^{-4t}\right]u(t)$

c) $f(t) = \left[5e^{-2t} + 10e^{-6t}\cos(8t - 53.13°)\right]u(t)$

d) $f(t) = \left[8 + 50e^{-7t}\cos(24t + 16.26°)\right]u(t)$

sol'n: a) We use the method of partial fractions to find inverse Laplace transforms that are the time-domain solutions to circuit problems.

With partial fractions, we write $F(s)$ as a sum of pole terms.

Here, we write $F(s) = \dfrac{K_1}{s+1} + \dfrac{K_2}{s+2} + \dfrac{K_3}{s+3}$

Comment: Partial fractions are another example of changing bases. We use the poles $\dfrac{1}{s+a}$ as the basis functions, and we have coefficients $K_1, K_2, \ldots$ multiplying the basis functions. Compare this with Fourier series where the basis functions are cos( ) and sin( ).

To find K_m for a pole term of form $\dfrac{K_m}{s+a}$,

we use $K_m = F(s)(s+a)\Big|_{s=-a}$.

To see why this works, consider what happens if we have $F(s)$ already in partial fraction form:

$$F(s)(s+a_1)\Big|_{s=-a_1} = \frac{K_1}{s+a_1}(s+a_1) + \frac{K_2}{s+a_2}(s+a_1) + \frac{K_3}{s+a_3}(s+a_1)\Bigg|_{s=-a_1}$$

$$= K_1 + K_2\frac{(s+a_1)}{s+a_2} + K_3\frac{(s+a_1)}{s+a_3}\Bigg|_{s=-a_1}$$

$$= K_1 + K_2 \cdot \frac{0}{-a_1+a_2} + K_3 \cdot \frac{0}{-a_1+a_3}$$

$$= K_1$$

Here, we have:

$$K_1 = \frac{18s^2+66s+54}{\cancel{(s+1)}(s+2)(s+3)}\cancel{(s+1)}\Bigg|_{s=-1} = \frac{18-66+54}{(-1+2)(-1+3)}$$

$$= \frac{6}{2} = 3$$

$$K_2 = \frac{18s^2+66s+54}{(s+1)\cancel{(s+2)}(s+3)}\cancel{(s+2)}\Bigg|_{s=-2} = \frac{18\cdot 4-66\cdot 2+54}{(-2+1)(-2+3)}$$

$$= \frac{-6}{-1} = 6$$

$$K_3 = \frac{18s^2+66s+54}{(s+1)(s+2)\cancel{(s+3)}}\cancel{(s+3)}\Bigg|_{s=-3} = \frac{18\cdot 9-66\cdot 3+54}{(-3+1)(-3+2)}$$

$$= \frac{18}{2} = 9$$

$$F(s) = \frac{3}{s+1} + \frac{6}{s+2} + \frac{9}{s+3}$$

Now we use $\mathcal{L}^{-1}\left\{\frac{k}{s+a}\right\} = ke^{-at}u(t)$.

$$f(t) = \left[3e^{-t} + 6e^{-2t} + 9e^{-3t}\right]u(t)$$

b) $$F(s) = \frac{K_1}{\underbrace{s}_{s+0}} + \frac{K_2}{s+2} + \frac{K_3}{s+4}$$

$$K_1 = F(s)\,s\Big|_{s=0} = \frac{25\cdot\overset{s^2}{0} + 86\cdot\overset{s}{0} + 40}{\cancel{s}\,(\underset{s}{0}+2)(\underset{s}{0}+4)}\,\cancel{s} = \frac{40}{8} = 5$$

$$K_2 = \frac{25s^2 + 86s + 40}{s\cancel{(s+2)}(s+4)}\cancel{(s+2)}\Bigg|_{s=-2} = \frac{25\cdot 4 - 86\cdot 2 + 40}{-2(-2+4)} = 8$$

$$K_3 = \frac{25s^2 + 86s + 40}{s(s+2)\cancel{(s+4)}}\cancel{(s+4)}\Bigg|_{s=-4} = \frac{25\cdot 16 - 86\cdot 4 + 40}{-4(-4+2)} = 12$$

$$F(s) = \frac{5}{s} + \frac{8}{s+2} + \frac{12}{s+4} \qquad \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = u(t)$$

$$\therefore\ f(t) = \left[5 + 8e^{-2t} + 12e^{-4t}\right]u(t)$$

c) We need to find poles for denominator term $s^2 + 12s + 100$

$$s^2 + 12s + 100 = 0 \quad\Rightarrow\quad s_{1,2} = -\frac{12}{2} \pm \sqrt{\left(\frac{12}{2}\right)^2 - 100}$$

$$= -6 \pm j8$$

$$s^2 + 12s + 100 = (s - s_1)(s - s_2)$$

$$\therefore\ F(s) = \frac{11s^2 + 172s + 700}{(s+2)(s+6+j8)(s+6-j8)}$$

$$F(s) = \frac{K_1}{s+2} + \frac{K_2}{s+6+j8} + \frac{K_2^*}{s+6-j8}$$

Because the poles for $s^2+12s+100$ are a conjugate pair, the coefficients are a conjugate pair: K_2, K_2^*. Thus, we only need to find K_2, not a K_3.

$$K_1 = \frac{11s^2+172s+700}{\cancel{(s+2)}(s^2+12s+100)}\cancel{(s+2)}\Bigg|_{s=-2} = \frac{11\cdot 4 - 172\cdot 2 + 700}{4 - 12\cdot 2 + 100} = 5$$

$$K_2 = \frac{11s^2+172s+700}{(s+2)\cancel{(s+6+j8)}(s+6-j8)}\cancel{(s+6+j8)}\Bigg|_{s=-6-j8}$$

$$= \frac{11\cdot[6^2 + j2\cdot 6\cdot 8 - 8^2] + 172(-6-j8) + 700}{(-6-j8+2)(-j16)}$$

$$= \frac{11(-28+j96) - 172(6+j8) + 700}{(-4-j8)(-j16)}$$

$$= \frac{-308 + j1056 - 1032 - j1376 + 700}{j64 - 128}$$

$$= \frac{-640 - j320}{-128 + j64} = 3 + j4$$

$$\therefore\ F(s) = \frac{5}{s+2} + \frac{3+j4}{s+6+j8} + \frac{3-j4}{s+6-j8}$$

Now we convert the conjugate pole pair terms back to denominator $s^2+12s+100$ so we can use inverse Laplace transforms that take us back to $e^{-at}\sin(\omega t)\,u(t)$ and $e^{-at}\cos(\omega t)\,u(t)$.

$$\frac{3+j4}{s+6+j8} + \frac{3-j4}{s+6-j8} = \frac{(3+j4)(s+6-j8) + (3-j4)(s+6+j8)}{s^2+12s+100}$$

$$= \frac{2\cdot 3s + 2(3\cdot 6 + 4\cdot 8)}{s^2 + 12s + 100} = \frac{6s + 100}{s^2 + 12s + 100}$$

In general $\dfrac{K}{s+\alpha+j\beta} + \dfrac{K^*}{s+\alpha-j\beta}$

$$= \frac{2\,\mathrm{Re}[K]\,s + 2\left(\mathrm{Re}[K]\cdot\alpha + \mathrm{Im}[K]\cdot\beta\right)}{(s+\alpha)^2 + \beta^2}$$

Now we observe that

$$\mathcal{L}\{f(t)\} = \mathcal{L}\left\{c_1 e^{-at}\cos(\omega t)\,u(t)\right\} + \mathcal{L}\left\{c_2 e^{-at}\sin(\omega t)\,u(t)\right\}$$

$$= \frac{c_1(s+a)}{(s+a)^2+\omega^2} + \frac{c_2\,\omega}{(s+a)^2+\omega^2} = \frac{c_1 s + c_1 a + c_2\omega}{(s+a)^2+\omega^2}$$

$$= \frac{c_1 s + (c_1 a + c_2\omega)}{s + 2as + (a^2+\omega^2)}$$

Now we equate terms, first in the denominator, then in the numerator.

First, $2as = 12s$ *in denominator* $\Rightarrow a = 6$,
($a = -$*damping factor* $= -\mathrm{Re}[pole]$).

Second, $a^2+\omega^2 = 100$ *in denominator* $\Rightarrow \omega = 8$,
($\omega = \sqrt{100-a^2}$ = *frequency* $= |\mathrm{Im}[pole]|$).

Third, $c_1 s = 6s$ *in numerator* $\Rightarrow c_1 = 6$,
(*not function of* a & ω).

Fourth, $c_1 a + c_2\omega = 100$ *in numerator* $\Rightarrow c_2 = 8$
$\left(c_2 = \dfrac{100 - c_1 a}{\omega}\right)$.

term from single pole term $\frac{5}{s+2}$ earlier

$$\therefore f(t) = \left[5e^{-2t} + 6e^{-6t}\cos(\omega t) + 8e^{-6t}\sin(\omega t)\right]u(t)$$

In general, we have $\dfrac{bs+d}{(s+a)^2+\omega^2}$ for $F(s)$,

and we have $c_1 = b$, $c_2 = \dfrac{d - b\cdot a}{\omega}$.

Finally, we can convert our $f(t)$ from rectangular form to polar form.

$$6\cos(\omega t) + 8\sin(\omega t) = \sqrt{6^2+8^2}\cos\left(\omega t + \tan^{-1}\frac{-8}{6}\right)$$

because $8\sin(\omega t)$ has phasor $-j8$

$$\text{"} \quad = 10\cos(\omega t - 53.13°)$$

$$\therefore \quad f(t) = \left[5e^{-2t} + 10\cos(\omega t - 53.13°)\right]u(t)$$

d) $$F(s) = \frac{56s^2+112s+5000}{s(s^2+14s+625)} = \frac{56s^2+112s+5000}{s(s+7+j24)(s+7-j24)}$$

$$= \frac{K_1}{s} + \frac{K_2}{s+7+j24} + \frac{K_2^*}{s+7-j24}$$

$$K_1 = \left.\frac{56s^2+112s+5000}{\cancel{s}(s^2+14s+625)}\cancel{s}\right|_{s=0} = \frac{5000}{625} = 8$$

$$K_2 = \left.\frac{56s^2+112s+5000}{s\,\cancel{(s+7+j24)}(s+7-j24)}\cancel{(s+7+j24)}\right|_{s=-7-j24}$$

$$= \frac{56\left[(7^2-24^2)+j2\cdot7\cdot24\right] + 112(-7-j24) + 5000}{(-7-j24)(-j2\cdot24)}$$

$$= \frac{-29512 + j18816 - 784 - j2688 + 5000}{1152 - j336} = \frac{-25296 + j16128}{-1152 + j336}$$

$$= 24 - j7 \qquad = 25\angle{-16.26°}$$

Now we use $\mathcal{L}\left\{2ce^{-at}\cos(\omega t - \theta)\cdot u(t)\right\} = \dfrac{c\angle\theta}{s+a+j\omega} + \dfrac{c\angle{-\theta}}{s+a-j\omega}$

or $\mathcal{L}\left\{2|K|e^{-at}\cos(\omega t - \angle K)\,u(t)\right\} = \frac{K}{s+a+j\omega} + \frac{K^*}{s+a-j\omega}$

Thus, we can find the polar form of $f(t)$ directly from K.

$$\therefore\ f(t) = \left[8 + 50e^{-7t}\cos(24t + 16.26°)\right]u(t)$$