Find $f(t) = \mathcal{L}^{-1}\{F(s)\}$ for a) $F(s) = \dfrac{20s^2}{(s+1)^3}$

b) $F(s) = \dfrac{5(s+2)^2}{s^4(s+1)}$

ans: a) $f(t) = (10t^2 - 40t + 20)e^{-t}\,u(t)$

b) $f(t) = \left(5e^{-5t} + \frac{10}{3}t^3 + 5t - 5\right)u(t)$

soln: a) Using partial fractions, we write

$$F(s) = \frac{K_1}{(s+1)^3} + \frac{K_2}{(s+1)^2} + \frac{K_3}{s+1}$$

Now we observe that, if we can indeed write $F(s)$ this way, then we have:

$$F(s)(s+1)^3 = K_1 + K_2(s+1) + K_3(s+1)^2$$

$$\text{Thus, } F(s)(s+1)^3\Big|_{s=-1} = K_1 + K_2 \cdot 0 + K_3 \cdot 0^2 = K_1$$

To find K_2, we differentiate:

$$\frac{d}{ds}\left[F(s)(s+1)^3\right] = K_2 + 2K_3(s+1)$$

$$\text{Thus, } \frac{d}{ds}\left[F(s)(s+1)^3\right] = K_2 + 2K_3 \cdot 0 = K_2$$

To find K_3, differentiate again:

$$\frac{d^2}{ds^2}\left[F(s)(s+1)^3\right] = 2K_3$$

(Divide by 2 to get K_3)

This came from $(s+1)^2$ and leads to factorial multipliers (e.g. 3·2·1) for high-order poles $(s+a)^n$.

In general, if we have $F(s) = \dfrac{K_1}{(s+a)^n} + \dfrac{K_2}{(s+a)^{n-1}} + \dots + \dfrac{K_n}{s+a}$,

then $K_m = \dfrac{1}{(m-1)!}\left(\dfrac{d^{m-1}}{ds^{m-1}}\left[F(s)(s+a)^n\right]\right)\Bigg|_{s=-a}$.

Here, we have $F(s)(s+1)^3 = \dfrac{20s^2}{(s+1)^3}(s+1)^3 = 20s^2$.

$$K_1 = 20s^2\Big|_{s=-1} = 20$$

$$K_2 = \frac{d}{ds}20s^2\Big|_{s=-1} = 40s\Big|_{s=-1} = -40$$

$$K_3 = \frac{1}{2}\frac{d^2}{ds^2}20s^2\Big|_{s=-1} = \frac{1}{2}\frac{d}{ds}40s\Big|_{s=-1} = \frac{1}{2}\cdot 40 = 20$$

$$\therefore\ F(s) = \frac{20}{(s+1)^3} - \frac{40}{(s+1)^2} + \frac{20}{s+1}$$

Now we use $\mathcal{L}^{-1}\left\{\dfrac{k}{(s+a)^n}\right\} = \dfrac{k}{(n-1)!}t^{n-1}e^{-at}u(t)$

Note: we get another factorial term, again arising from polynomial differentiation: $\dfrac{d}{ds}(s+a)^n = n(s+a)^{n-1}$, $\dfrac{d^2}{ds^2}(s+a)^n = n(n-1)(s+a)^{n-2}$, etc.

The differention comes from the operational transform $\mathcal{L}\{tf(t)\} = -\dfrac{d}{ds}F(s)$, where $f(t) = t^{n-1}e^{-at}u(t)$.

Thus, $f(t) = \left[\dfrac{20}{2!}t^2e^{-t} - \dfrac{40}{1!}te^{-t} + \dfrac{20}{0!}e^{-t}\right]u(t)$

$\uparrow = 1$

or $f(t) = (10t^2 - 40t + 20)e^{-t}u(t)$

b) Same method as in (a).

$$F(s) = \frac{K_1}{s^4} + \frac{K_2}{s^3} + \frac{K_3}{s^2} + \frac{K_4}{s} + \frac{K_5}{s+1}$$

$s+a$ where $a=0$

$$F(s)\,s^4 = \frac{5(s+2)^2}{s+1}$$

$$K_1 = F(s)\,s^4\Big|_{s=0} = \frac{5(0+2)^2}{0+1} = 20$$

$$K_2 = \frac{d}{ds}\left[F(s)\,s^4\right]\Big|_{s=0} = \frac{d}{ds}\frac{5(s+2)^2}{s+1}\Big|_{s=0}$$

$$\text{or } K_2 = \frac{d}{ds}5(s+2)^2(s+1)^{-1}\Big|_{s=0} = 5\cdot 2\cdot(s+2)(s+1)^{-1} + 5(s+2)^2(-1)(s+1)^{-2}\Big|_{s=0}$$

$$\text{or } K_2 = 5\left[\frac{2(s+2)(s+1) - (s+2)^2}{(s+1)^2}\right]\Big|_{s=0} = \frac{5\left[2\cdot 2\cdot 1 - 2^2\right]}{1^2}$$

$$\text{or } K_2 = 0$$

$$K_3 = \frac{1}{2}\frac{d^2}{ds^2}\frac{5(s+2)^2}{s+1} = \frac{d}{ds}5\left[\frac{2(s+2)(s+1) - (s+2)^2}{(s+1)^2}\right]\Big|_{s=0}$$

$$" = \frac{1}{2}\frac{d}{ds}5\frac{(s+2)}{(s+1)^2}\left[2(s+1) - (s+2)\right]\Big|_{s=0} = \frac{d}{ds}5\frac{(s+2)\,s}{(s+1)^2}\Big|_{s=0}$$

$$" = \frac{5}{2}\frac{d}{ds}(s+2)\,s\,(s+1)^{-2} = 5\left[(2s+2)(s+1)^{-2} + (s^2+2s)(-2)(s+1)^{-3}\right]_{s=0}$$

$$K_3 = \frac{5}{2}\left[2\cdot 1^{-2} + 0\cdot(-2)(1)^{-3}\right] = \frac{10}{2}$$

$$K_3 = 5$$

$$K_4 = \frac{1}{3!}\frac{d^3}{ds^3}\frac{5(s+2)^2}{s+1}\Bigg|_{s=0} = \frac{d}{ds}\left[(2s+2)(s+1)^{-2} + (s^2+2s)(-2)(s+1)^{-3}\right]\Bigg|_{s=0}$$

$$= \frac{5}{6}\,2(s+1)^{-2} + (2s+2)(-2)\cdot 1 + (2s+2)(-2)(s+1)^{-3} + (s^2+2s)(-2)(-3)(s+1)^{-4}\Bigg|_{s=0}$$

$$= \frac{5}{6}\,2\cdot 1^{-2} + 2(-2)\cdot 1 + 2(-2)\,1^{-3} + 0(-2)(-3)\,1^{-4}$$

$$= \frac{5}{6}(2-4-4)$$

$$K_4 = \frac{5(-6)}{6} = -5$$

$$K_5 = F(s)(s+1)\Big|_{s=-1} \qquad \text{single pole at } s=-1$$

$$= \frac{5(s+2)^2}{s^4}\Bigg|_{s=-1} = \frac{5\cdot 1^2}{(-1)^4}$$

$$K_5 = 5$$

$$\therefore\ F(s) = \frac{20}{s^4} + \frac{0}{s^3} + \frac{5}{s^2} + \frac{-5}{s} + \frac{5}{s+1}$$

use $\mathcal{L}^{-1}\left\{\frac{k}{(s+a)^n}\right\} = \frac{k}{(n-1)!}t^{n-1}e^{-at}u(t) \qquad a=0 \text{ for } \frac{k}{s^n}$

$$f(t) = \left\{\left(\frac{20}{6}t^3 + 5t - 5\right)e^{0t} + 5e^{-t}\right\}u(t)$$

or $f(t) = \left\{5e^{-t} + \frac{10}{3}t^3 + 5t - 5\right\}u(t)$