EX: Find the inverse Laplace transform for the following expression:

$$F(s) = -\frac{s^2 + 2s + 3}{(s+1)(s^2 + 4s + 5)}$$

SOL'N: We may solve this problem using standard partial fraction techniques where we write $F(s)$ in terms of root (or pole) terms:

$$F(s) = \frac{K_1}{s+1} + \frac{K_2}{s+2+j} + \frac{K_2^*}{s+2-j}$$

An alternative approach that requires less work, however, involves choosing convenient values of s for which we evaluate $F(s)$ in its original form and in the following form (in which we have written the complex root terms as a sum of terms that are a decaying cosine and sine in the time domain):

$$F(s) = \frac{K_1}{s+1} + \frac{K_2(s+a) + K_3(\omega)}{(s+a)^2 + \omega^2}$$

For the values of our root terms, $a = 2$ and $\omega = -1$:

$$F(s) = \frac{K_1}{s+1} + \frac{K_2(s+2) + K_3(1)}{(s+2)^2 + 1^2}$$

Note that one may also use the preceding standard partial fraction expansion of $F(s)$ with the method that follows. The key idea is to find coefficients in whatever expansion we choose by equating $F(s)$ at several values of s.

Choosing convenient values of s is a matter of personal preference, but we must avoid choosing root (or pole) values. Here, we consider using $s = 0$, $s = 1$, and $s = -a = -2$.

$$F(0) = -\frac{0^2 + 2\cdot 0 + 3}{(0+1)(0^2 + 4\cdot 0 + 5)} = \frac{K_1}{0+1} + \frac{K_2(0+2) + K_3(1)}{(0+2)^2 + 1^2}$$

or

$$F(0) = -\frac{3}{(1)(5)} = \frac{K_1}{1} + \frac{K_2\cdot 2 + K_3\cdot 1}{5}$$

$$F(1) = -\frac{1^2 + 2\cdot 1 + 3}{(1+1)(1^2 + 4\cdot 1 + 5)} = \frac{K_1}{1+1} + \frac{K_2(1+2) + K_3(1)}{(1+2)^2 + 1^2}$$

or

$$F(1) = -\frac{6}{(2)(10)} = \frac{K_1}{2} + \frac{K_2 \cdot 3 + K_3 \cdot 1}{10}$$

$$F(-2) = -\frac{(-2)^2 + 2 \cdot (-2) + 3}{(-2+1)((-2)^2 + 4(-2) + 5)}$$

and

$$F(-2) = \frac{K_1}{-2+1} + \frac{K_2(-2+2) + K_3(1)}{(-2+2)^2 + 1^2}$$

or

$$F(-2) = -\frac{3}{-1} = \frac{K_1}{-1} + \frac{K_3(1)}{1}$$

We now have three equations in three unknowns. Putting each equation over its common denominator and then multiplying both sides by that denominator allows us to write these equations in a simple form:

$$-3 = 5K_1 + 2K_2 + K_3$$
$$-3 = 5K_1 + 3K_2 + K_3$$
$$3 = -K_1 + K_3$$

From the first and second equations we see that $K_2 = 0$. Solving the second and third equations then yields the values of the other two coefficients:

$$K_1 = -1 \qquad K_2 = 0 \qquad K_3 = 2$$

Thus, we have found the coefficients for the original form of $F(s)$:

$$F(s) = \frac{K_1}{s+1} + \frac{K_2(s+2)}{(s+2)^2 + 1^2} + \frac{K_3(1)}{(s+2)^2 + 1^2}$$

In the time domain we have three corresponding terms:

$$f(t) = K_1 e^{-t} + K_2 e^{-2t}\cos(t) + K_3 e^{-2t}\sin(t)$$
$$f(t) = -e^{-t} + 2e^{-2t}\sin(t)$$