Find Laplace Transforms:

a) $f(t) = t e^{-at}$

b) $f(t) = \sin \omega t$

c) $f(t) = \sin(\omega t + \theta)$

d) $f(t) = \cosh t$

e) $f(t) = \cosh(t+\theta)$

ans:

a) $\dfrac{1}{(s+a)^2}$

b) $\dfrac{\omega}{s^2+\omega^2}$

c) $\dfrac{\omega \cos\theta + s\sin\theta}{s^2+\omega^2}$

d) $\dfrac{s}{s^2-1}$

e) $\dfrac{\sinh\theta + s\cosh\theta}{s^2-1}$

sol'n: a)

- From table on inside front cover of text we have $\mathcal{L}\{te^{-at}\} = \dfrac{1}{(s+a)^2}$. (Easy sol'n)

- Suppose we only had the list of operational transforms and knew $\mathcal{L}\{e^{-at}\} = \dfrac{1}{s+a}$.

  In that case, we could use $\mathcal{L}\{tf(t)\} = -\dfrac{d F(s)}{ds}$, where $f(t) = e^{-at}$ and $F(s) = \dfrac{1}{s+a}$.

$$-\frac{dF(s)}{ds} = -\frac{d}{ds}\frac{1}{s+a} = -\frac{d}{ds}(s+a)^{-1} = -(-1)(s+a)^{-2}$$

$$= \frac{1}{(s+a)^2} \checkmark$$

- Suppose we only had the list of operational transforms and knew $\mathcal{L}\{t\} = \dfrac{1}{s^2}$.

  In that case, we could use $\mathcal{L}\{e^{-at}f(t)\} = F(s+a)$, where $f(t) = t$ and $F(s) = \dfrac{1}{s^2}$.

$$F(s+a) = \frac{1}{(s+a)^2} \checkmark$$

- Suppose we only had the basic definition of the Laplace transform and no tables at all.

In that case, we have $\mathcal{L}\{te^{-at}\} = \int_{0^-}^{\infty} te^{-at}e^{-st}\,dt$

$$= \int_{0^-}^{\infty} te^{-(a+s)t}\,dt = e^{-(a+s)t}\left[\frac{t}{-(a+s)} - \frac{1}{(a+s)^2}\right]\Bigg|_{0^-}^{\infty}$$

(above from integral table p. 1009)

$$= \underbrace{e^{-\infty}\cdot\left(\frac{\infty}{-(a+s)} - 1\right)} - e^{-(a+s)\cdot 0}\left[\frac{0^-}{-(a+s)} - \frac{1}{(a+s)^2}\right]$$

Assume this term $= 0$ because $e^{-(a+s)t} \to 0$ as $t \to \infty$ and does so faster than $t \to \infty$ so $\lim_{t\to\infty} e^{-(a+s)t}\cdot\frac{t}{-(a+s)} = 0$. To prove this, use $e^{-(a+s)t} = \frac{1}{e^{(a+s)t}} = \frac{1}{1+(a+s)t+\frac{(a+s)t^2}{2!}+\ldots}$ and we see that the denominator gets large faster than the factor of $\frac{t}{-(a+s)}$ that $e^{-(a+s)t}$ is multiplied by.

$$= -e^{0^-}\cdot\left[\frac{-1}{(a+s)^2}\right] = \frac{(-1)(-1)}{(a+s)^2} = \frac{1}{(a+s)^2} = \frac{1}{(s+a)^2} \checkmark$$

b) $\mathcal{L}\{f(t) = \sin \omega t\} = \int_{0^-}^{\infty} \sin \omega t\, e^{-st}\,dt$

Now use identity $\sin \omega t = \dfrac{e^{j\omega t} - e^{-j\omega t}}{j2}$

$$= \int_{0^-}^{\infty} \frac{e^{j\omega t}e^{-st}}{j2}\,dt - \int_{0^-}^{\infty} \frac{e^{-j\omega t}e^{-st}}{j2}\,dt$$

$$= \frac{1}{j2}\int_{0^-}^{\infty} e^{(j\omega - s)t}\,dt - \frac{1}{j2}\int_{0^-}^{\infty} e^{-(j\omega + s)t}\,dt$$

$$= \frac{1}{j2}\,\frac{e^{(j\omega - s)t}}{j\omega - s}\Bigg|_{0^-}^{\infty} - \frac{1}{j2}\,\frac{e^{-(j\omega+s)t}}{-(j\omega+s)}\Bigg|_{0^-}^{\infty}$$

(Note that $\int e^{ax} = \frac{e^{ax}}{a}$ whether a is real or complex.)

$$= \frac{1}{j2}\left[\cancelto{0}{\frac{e^{(j\omega-s)\infty}}{j\omega-s}} - \frac{e^{(j\omega-s)0^-}}{j\omega-s}\right] - \frac{1}{j2}\left[\cancelto{0}{\frac{e^{-(j\omega+s)\infty}}{-(j\omega+s)}} - \frac{e^{-(j\omega+s)0^-}}{-(j\omega+s)}\right]$$

(As usual, we assume we get 0 when $t=\infty$. What we really assume is that the real part of s is sufficiently large and positive enough that
$$|e^{-st}| = |e^{-Re[s]t}||e^{-j\,Im[s]t}| = |e^{-Re[s]t}|\cdot 1 \to 0 \text{ as } t\to\infty.)$$

$$= \frac{1}{j2}\left(\frac{-1}{j\omega-s}\right) - \frac{1}{j2}\,\frac{-1}{-(j\omega+s)} = \frac{1}{j2}\left(\frac{-1}{j\omega-s} - \frac{1}{j\omega+s}\right)$$

$$= \frac{1}{j2}\,\frac{-(j\omega+\cancel{s}) - (j\omega-\cancel{s})}{(j\omega-s)(j\omega+s)} = -\frac{\cancel{2}}{\cancel{j2}}\,\frac{\cancel{j}\omega}{-\omega^2-s^2}$$

$= \frac{\omega}{s^2+\omega^2}$. From table on inside front cover we also have $\mathcal{L}\{\sin \omega t\} = \frac{\omega}{s^2+\omega^2}$. ✓

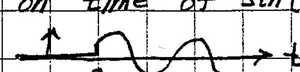
c) $\mathcal{L}\{f(t) = \sin(\omega t + \theta)\}$

Warning! One might be tempted to try to use the operational transform:
$\mathcal{L}\{f(t-a)u(t-a),\ a>0\} = e^{-as}F(s)$.

But if we write $\sin(\omega t+\theta) = \sin\left(\omega\left(t+\frac{\theta}{\omega}\right)\right)$ and use $a = -\frac{\theta}{\omega}$ for the above operational transform we have 2 problems:

1) $a = -\frac{\theta}{\omega}$ might be < 0, and we need $a>0$.

2) We don't have $u\left(t+\frac{\theta}{\omega}\right)$ multiplying $\sin\left(\omega\left(t+\frac{\theta}{\omega}\right)\right)$. The step function is essential, because it delays the turn on time of $\sin\left(\omega\left(t+\frac{\theta}{\omega}\right)\right)$ until time $a = -\frac{\theta}{\omega}$:



So we can only apply $\mathcal{L}\{f(t-a)u(t-a)\} = e^{-as}F(s)$ when we have signals with delayed turn on. Conversely, if we do have a delayed turn on, then we rewrite our function so it is of form $f(t-a)$, and we must use this operational transform. This is illustrated in another problem.

For $\mathcal{L}\{\sin(\omega t+\theta)\}$ we use a trigonometric identity:

$$\mathcal{L}\{\sin(\omega t+\theta)\} = \mathcal{L}\{\sin(\omega t)\overbrace{\cos\theta}^{\text{const}} + \cos(\omega t)\overbrace{\sin\theta}^{\text{const}}\}$$

$$\mathcal{L}\{\sin(\omega t)\} = \frac{\omega}{s^2+\omega^2} \qquad \mathcal{L}\{\cos(\omega t)\} = \frac{s}{s^2+\omega^2} \quad \text{(from tables)}$$

$$\therefore \mathcal{L}\{\sin(\omega t+\theta)\} = \cos\theta \cdot \frac{\omega}{s^2+\omega^2} + \sin\theta \frac{s}{s^2+\omega^2}$$

(Note that $\cos\theta$ and $\sin\theta$ are just constants and we can use $\mathcal{L}\{k f(t)\} = k\mathcal{L}\{f(t)\}$ for k constant.)

$$\mathcal{L}\{\sin(\omega t+\theta)\} = \frac{\cos(\theta)\cdot\omega + \sin(\theta)\cdot s}{s^2+\omega^2}.$$

d) $$\mathcal{L}\{f(t) = \cosh t\} = \int_{0^-}^{\infty} \underbrace{\frac{e^{t}+e^{-t}}{2}}_{\equiv \cosh t} e^{-st}\,dt$$

$$= \frac{1}{2}\int_{0^-}^{\infty} e^{(1-s)t}\,dt + \frac{1}{2}\int_{0^-}^{\infty} e^{-(1+s)t}\,dt$$

$$= \frac{1}{2}\left.\frac{e^{(1-s)t}}{1-s}\right|_{0^-}^{\infty} + \frac{1}{2}\left.\frac{e^{-(1+s)t}}{-(1+s)}\right|_{0^-}^{\infty}$$

$$= \frac{1}{2}\left[\underbrace{\frac{e^{-\infty}}{1-s}}_{0} - \frac{e^{0}}{1-s}\right] + \frac{1}{2}\left[\underbrace{\frac{e^{-\infty}}{-(1+s)}}_{0} - \frac{e^{0^-}}{-(1+s)}\right]$$

$$= \frac{1}{2}\left[\frac{-1}{1-s} - \frac{1}{-(1+s)}\right] = \frac{1}{2}\left[\frac{1}{s-1} + \frac{1}{s+1}\right]$$

$$= \frac{1}{2}\,\frac{s+1 + s-1}{(s-1)(s+1)} = \frac{2s}{2(s^2-1)} = \frac{s}{s^2-1}$$

e) $$\mathcal{L}\{\cosh(t+\theta)\} = \mathcal{L}\left\{\frac{e^{t+\theta} + e^{-(t+\theta)}}{2}\right\}$$

$$'' = \frac{1}{2}\mathcal{L}\{e^{t}e^{\theta}\} + \frac{1}{2}\mathcal{L}\{e^{-t}e^{-\theta}\} = \frac{1}{2}e^{\theta}\mathcal{L}\{e^{t}\} + \frac{1}{2}e^{-\theta}\mathcal{L}\{e^{-t}\}$$

$$\left(\mathcal{L}\{e^{t}\} = \frac{1}{s-1} \qquad \mathcal{L}\{e^{-t}\} = \frac{1}{s+1}\right)$$

$$'' = \frac{1}{2}e^{\theta}\frac{1}{s-1} + \frac{1}{2}e^{-\theta}\frac{1}{s+1} = \frac{1}{2}\,\frac{e^{\theta}(s+1) + e^{-\theta}(s-1)}{s^2-1}$$

$$'' = \frac{1}{2}\,\frac{s(e^{\theta}+e^{-\theta}) + (e^{\theta}-e^{-\theta})}{s^2+1} = \frac{1}{s^2-1}\left[s\,\underbrace{\frac{e^{\theta}+e^{-\theta}}{2}}_{\cosh\theta} + \underbrace{\frac{e^{\theta}-e^{-\theta}}{2}}_{\sinh\theta}\right]$$

$$'' = \frac{s\cosh\theta + \sinh\theta}{s^2-1}.$$