Neil E Cotter
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Linear Algebra – Invariants – Diagonalization

Tool: For any matrix A, $n \times n$ with n distinct eigenvalues,

$$\boxed{\begin{array}{l} A = S \Lambda S^{-1} \\ \text{where} \quad S = \begin{bmatrix} | & & | \\ \vec{\varphi}_1 & \cdots & \vec{\varphi}_n \\ | & & | \end{bmatrix} \quad \text{col's = eigenvectors}, \quad |\vec{\varphi}_i| = 1 \text{ normalized length (optional)} \\ \Lambda = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \quad \text{eigenvalues on diagonal} \end{array}}$$

pf: $A\vec{\varphi}_i = \lambda_i \vec{\varphi}_i$ so $AS = S\Lambda$ and $A = S\Lambda S^{-1}$.

ex: Compute $A^k = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^k$

soln: Diagonalize, $A^k = (S\Lambda S^{-1})^k = S\Lambda \cancel{S^{-1}} \cancel{S} \Lambda \cancel{S^{-1}} \cdots \cancel{S} \Lambda S^{-1} = \underline{\underline{S\Lambda^k S^{-1}}}$

and $\Lambda^k = \begin{bmatrix} \lambda_1^k & & 0 \\ & \ddots & \\ 0 & & \lambda_n^k \end{bmatrix} = \begin{bmatrix} \lambda_1^k & 0 \\ 0 & \lambda_2^k \end{bmatrix}$ since $n=2$

Find S and Λ. det $A - \lambda I = 0$ $\quad \begin{vmatrix} 1-\lambda & 1 \\ 1 & -1-\lambda \end{vmatrix} = 0$

$$(1-\lambda)\cdot(-1-\lambda) - 1 = \lambda^2 - 1 - 1 = 0$$

$$\lambda^2 - 2 = 0 \qquad \lambda = \pm\sqrt{2} \qquad \lambda_1 = \sqrt{2} \quad \lambda_2 = -\sqrt{2}$$

$\vec{\varphi}_1$: $\begin{bmatrix} 1-\sqrt{2} & 1 \\ 1 & -1-\sqrt{2} \end{bmatrix}\begin{bmatrix} 1 \\ x \end{bmatrix} = 0 \qquad -x = 1-\sqrt{2} \qquad \vec{\varphi}_1 = \dfrac{-1}{\sqrt{1^2+(1-\sqrt{2})^2}}\begin{bmatrix} -1 \\ 1-\sqrt{2} \end{bmatrix}$

$\vec{\varphi}_2$: $\begin{bmatrix} 1+\sqrt{2} & 1 \\ 1 & -1+\sqrt{2} \end{bmatrix}\begin{bmatrix} 1 \\ y \end{bmatrix} = 0 \qquad -y = 1+\sqrt{2} \qquad \vec{\varphi}_2 = \dfrac{-1}{\sqrt{1^2+(1+\sqrt{2})^2}}\begin{bmatrix} -1 \\ 1+\sqrt{2} \end{bmatrix}$

These are normalized to length 1.

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Linear Algebra – Invariants – Diagonalization (cont.)

$$\sqrt{1^2+(1-\sqrt{2})^2} = \sqrt{1+1-2\sqrt{2}+2} = \sqrt{4-2\sqrt{2}} = 2\sqrt{1-1/\sqrt{2}}$$

$$\sqrt{1^2+(1+\sqrt{2})^2} = \sqrt{1+1+2\sqrt{2}+2} = \sqrt{4+2\sqrt{2}} = 2\sqrt{1+1/\sqrt{2}}$$

$$\vec{\varphi}_1 = \frac{1}{2\sqrt{1-1/\sqrt{2}}}\begin{bmatrix} 1 \\ \sqrt{2}-1 \end{bmatrix} \qquad \vec{\varphi}_2 = \frac{1}{2\sqrt{1+1/\sqrt{2}}}\begin{bmatrix} 1 \\ -\sqrt{2}-1 \end{bmatrix}$$

If we do not normalize we ~~would~~ have:

$$\vec{\varphi}_1 = \begin{bmatrix} 1 \\ \sqrt{2}-1 \end{bmatrix} \qquad \varphi_2 = \begin{bmatrix} 1 \\ -\sqrt{2}-1 \end{bmatrix}$$

(we use these since they are less messy.)

$$\therefore \quad S = \begin{bmatrix} 1 & 1 \\ \sqrt{2}-1 & -\sqrt{2}-1 \end{bmatrix} \qquad \Lambda = \begin{bmatrix} \sqrt{2} & 0 \\ 0 & -\sqrt{2} \end{bmatrix}$$

$$A^k = S\Lambda^k S^{-1} = S\begin{bmatrix} \sqrt{2}^k & 0 \\ 0 & -\sqrt{2}^k \end{bmatrix} S^{-1}$$

check: $k=2$

$$A^2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = 2I$$

$$S^{-1} = \frac{1}{-\sqrt{2}\cancel{-1} - (\sqrt{2}\cancel{-1})}\begin{bmatrix} -\sqrt{2}-1 & -1 \\ 1-\sqrt{2} & 1 \end{bmatrix}$$

$$\text{since} \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 1 & 1 \\ \sqrt{2}-1 & -\sqrt{2}-1 \end{bmatrix}\underbrace{\begin{bmatrix} \sqrt{2}^2 & 0 \\ 0 & (-\sqrt{2})^2 \end{bmatrix}}_{=2I}\frac{-1}{2\sqrt{2}}\begin{bmatrix} -\sqrt{2}-1 & -1 \\ 1-\sqrt{2} & 1 \end{bmatrix}$$

$$= S\cdot 2I\cdot S^{-1}$$

$$= 2I\cdot S\cdot S^{-1} = 2I \checkmark \qquad \text{since} \quad S\cdot I = I\cdot S$$

Note: Diagonalization is useful for calculating A^k when k is very large. $A^k = S\Lambda^k S^{-1}$

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ex:

$$A = \begin{bmatrix} 1 & -1 \\ 3 & 5 \end{bmatrix}$$

$$\lambda_1 = 4 \quad \vec{\varphi}_1 = \begin{bmatrix} 1 \\ -3 \end{bmatrix} \qquad \lambda_2 = 2 \quad \vec{\varphi}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$A = S \Lambda S^{-1}$$

$$S = \begin{bmatrix} \begin{bmatrix} \vec{\varphi}_1 \end{bmatrix} & \begin{bmatrix} \vec{\varphi}_2 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -3 & -1 \end{bmatrix}$$

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}$$

$$S^{-1} = \frac{1}{2} \begin{bmatrix} -1 & -1 \\ 3 & 1 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} 1 & -1 \\ 3 & 5 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 1 & 1 \\ -3 & -1 \end{bmatrix}}_{S} \underbrace{\begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}}_{\Lambda} \underbrace{\begin{bmatrix} -1 & -1 \\ 3 & 1 \end{bmatrix} \cdot \frac{1}{2}}_{S^{-1}}$$

check by multiplying it out:

$$= \begin{bmatrix} 4 & 2 \\ -12 & -2 \end{bmatrix} \begin{bmatrix} -1 & -1 \\ 3 & 1 \end{bmatrix} \cdot \frac{1}{2}$$

$$= \begin{bmatrix} 2 & -2 \\ 6 & 10 \end{bmatrix} \cdot \frac{1}{2}$$

$$= \begin{bmatrix} 1 & -1 \\ 3 & 5 \end{bmatrix} \quad \checkmark$$

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Linear Algebra — Invariants — Diagonalization (cont.)

ex:

$$A = \begin{bmatrix} -5/2 & 1/2 \\ 1/2 & -5/2 \end{bmatrix} = S \Lambda S^{-1}$$

Find λ's from $\det A - \lambda I = 0$

$$\begin{vmatrix} -5/2 - \lambda & 1/2 \\ 1/2 & -5/2 - \lambda \end{vmatrix} = (-5/2 - \lambda)^2 - 1/2^2 = 0$$

$$(-5/2 - \lambda)^2 = 1/2^2 \qquad -5/2 - \lambda = \pm 1/2$$

$$\lambda_1 = -2 \qquad \lambda_2 = -3$$

Find eigvecs from $(A - \lambda_i I)\vec{\varphi}_i = \vec{0}$

$$\vec{\varphi}_1: \begin{bmatrix} -1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}\begin{bmatrix} 1 \\ x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad x = 1$$

$$\vec{\varphi}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\vec{\varphi}_2: \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}\begin{bmatrix} 1 \\ x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad x = -1$$

$$\vec{\varphi}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$S = \begin{bmatrix} | & | \\ \vec{\varphi}_1 & \vec{\varphi}_2 \\ | & | \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} & \begin{bmatrix} 1 \\ -1 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

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Linear Algebra - Invariants - Diagonalization (cont.)

Find S^{-1} from $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \dfrac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$$S^{-1} = -\frac{1}{2}\begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}$$

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix}$$

$$\therefore \underbrace{\begin{bmatrix} -5/2 & 1/2 \\ 1/2 & -5/2 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}}_{S}\underbrace{\begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix}}_{\Lambda}\underbrace{\begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}}_{S^{-1}}$$

$$A^k = S \quad \Lambda^k \quad S^{-1}$$

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Linear Algebra – Invariants – Diagonalization (cont.)

ex: $A = \begin{bmatrix} 5 & 3 \\ 3 & 13 \end{bmatrix}$ $\Lambda = \begin{bmatrix} 4 & 0 \\ 0 & 14 \end{bmatrix}$ $S = \begin{bmatrix} 3 & 1 \\ -1 & 3 \end{bmatrix}$ $S^{-1} = \frac{1}{10}\begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix}$

$$A = S\Lambda S^{-1} = \begin{bmatrix} 3 & 1 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 4 & 0 \\ 0 & 14 \end{bmatrix}\frac{1}{10}\begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 1 \\ -1 & 3 \end{bmatrix}\frac{1}{10}\begin{bmatrix} 12 & -4 \\ 14 & 42 \end{bmatrix}$$

$$= \frac{1}{10}\begin{bmatrix} 50 & 30 \\ 30 & 130 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 3 \\ 3 & 13 \end{bmatrix} \quad \checkmark$$