DEF: $M_{ij} \equiv$ minor matrix$_{ij}$ (of square matrix A) $\equiv A$ with row i and column j deleted.

EX:

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 3 & 6 \\ -5 & 1 & 4 \end{bmatrix}$$

$$M_{12} = \begin{bmatrix} 0 & 6 \\ -5 & 4 \end{bmatrix}$$

$$M_{32} = \begin{bmatrix} 1 & 2 \\ 0 & 6 \end{bmatrix}$$

TOOL: One method of computing the determinant of matrix A involves minors for one row or one column of a matrix.

$$|A| = \sum_j a_{ij}(-1)^{i+j}|M_{ij}| \text{ where } i \text{ is constant (i.e., one row)}$$

$$|A| = \sum_i a_{ij}(-1)^{i+j}|M_{ij}| \text{ where } j \text{ is constant (i.e., one column)}$$

EX:

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 3 & 6 \\ -5 & 1 & 4 \end{bmatrix}$$

Using minors of row one, we have

$$|A| = 1(-1)^{1+1}\left|\begin{bmatrix} 3 & 6 \\ 1 & 4 \end{bmatrix}\right| + 0\cdot(-1)^{1+2}\left|\begin{bmatrix} 0 & 6 \\ -5 & 4 \end{bmatrix}\right| + 2(-1)^{1+3}\left|\begin{bmatrix} 0 & 3 \\ -5 & 1 \end{bmatrix}\right|.$$

We apply the determinant calculation recursively to the 2 x 2 matrices (using the minors of the top row of each one):

$$\begin{aligned} |A| = \quad & 1\cdot\left(3(-1)^{1+1}\cdot\left|[4]\right| + 6(-1)^{1+2}\cdot\left|[1]\right|\right) \\ & -0\cdot\left(0(-1)^{1+1}\cdot\left|[4]\right| + 6(-1)^{1+2}\cdot\left|[-5]\right|\right) \\ & +2\cdot\left(0(-1)^{1+1}\cdot\left|[1]\right| + 3(-1)^{1+2}\cdot\left|[-5]\right|\right) \end{aligned}$$

NOTE: The power of –1 for the new, smaller minor matrices is determined by the index within that minor matrix rather than the index within the original matrix.

$$\begin{aligned}|A| = &\quad 1\cdot(3\cdot 4 - 6\cdot 1)\\ &-0\cdot(0\cdot 4 - 6\cdot(-5))\\ &+2\cdot(0\cdot 1 - 3\cdot(-5))\end{aligned}$$

$$\begin{aligned}|A| = &\quad 1\cdot(6)\\ &-0\cdot(30)\\ &+2\cdot(15)\end{aligned}$$

$$|A| = 36$$

NOTE: We may use minors for a row or column containing many entries equal to zero to reduce calculations.