28 Mar 1990 Linear Algebra – Dot product

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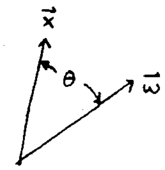
dot product (or inner product): $\vec{w} \circ \vec{x} = w_0 x_0 + \ldots + w_n x_n$



" $= |\vec{w}||\vec{x}| \cos\theta$

$\theta \equiv$ angle between $\vec{w}$ and $\vec{x}$

length (or absolute value): $|\vec{x}| = \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2} = \sqrt{\vec{x} \circ \vec{x}}$

length squared: $|\vec{x}|^2 = \vec{x} \circ \vec{x} = x_1^2 + \ldots + x_n^2$

Euclidean distance squared: $|\vec{w} - \vec{x}|^2 = (\vec{w} - \vec{x}) \circ (\vec{w} - \vec{x})$

Commutative: $\vec{w} \circ \vec{x} = \vec{x} \circ \vec{w}$

Distributive: $\vec{w} \circ (\vec{x} + \vec{y}) = \vec{w} \circ \vec{x} + \vec{w} \circ \vec{y}$

moral: "$\circ$" similar to mult.

Scaling: $a\vec{x} \circ b\vec{y} = ab(\vec{x} \circ \vec{y})$ $\quad a\vec{x} \equiv (ax_1, \ldots, ax_n)$

Orthogonality: $\vec{w} \circ \vec{x} = 0 \Rightarrow \vec{w} \perp \vec{x}$ since $\cos\theta = 0$

perpendicular

Norms: $|\vec{x}|_p = \sqrt[p]{|x_1|^p + \ldots + |x_n|^p}$ (p^{th} root)

$|\vec{x}|_1 = |x_1| + \ldots + |x_n|$ city blocks dist.

Note: $|\vec{x}| \neq |\vec{x}|_1$

$|\vec{x}| = |\vec{x}|_2$

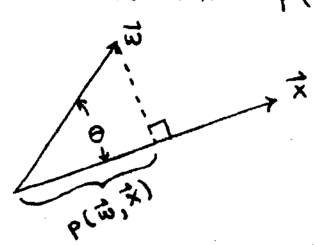
$|\vec{x}|_2 = \sqrt{x_1^2 + \ldots + x_n^2} = |\vec{x}|$ assume real #

so $|x_i|^2 = x_i^2$

$|\vec{x}|_\infty = \max_{x_i} |x_i|$ biggest element in vec.

Projections: (of $\vec{w}$ onto $\vec{x}$) $p(\vec{w}, \vec{x}) = \underbrace{\left(\vec{w} \circ \frac{\vec{x}}{|\vec{x}|}\right)}_{\text{scalar}} \underbrace{\frac{\vec{x}}{|\vec{x}|}}_{\text{unit vector in } \vec{x} \text{ directic}} \neq p(\vec{x}, \vec{w})$

= portion of $\vec{w}$ in $\vec{x}$ direction



$= \left(|\vec{w}| \cos\theta\right) \frac{\vec{x}}{|\vec{x}|}$

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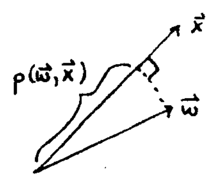
ex: $\vec{w} = (1,2,3)$ $\vec{x} = (4,5,6)$ $\vec{w} \circ \vec{x} = 1\cdot 4 + 2\cdot 5 + 3\cdot 6 = 32$

$$|\vec{w}| = \sqrt{1^2+2^2+3^2} = \sqrt{14} \qquad \cos\theta = \frac{32}{\sqrt{14}\sqrt{77}}, \quad \theta = 12^\circ$$
$$|\vec{x}| = \sqrt{4^2+5^2+6^2} = \sqrt{77}$$

$$\text{distance}^2 = |\vec{w} - \vec{x}|^2 = |(1-4,\ 2-5,\ 3-6)|^2 = |(-3,-3,-3)|^2$$
$$= (-3)^2 + (-3)^2 + (-3)^2 = 27$$

use projections to decompose $\vec{w}$ into components $\perp$ to $\vec{x}$ and $\parallel$ (parallel) to $\vec{x}$:



soln: $p(\vec{w},\vec{x})$ is part of $\vec{w}$ in $\vec{x}$ direction
The rest of $\vec{w}$ is $\therefore \perp \vec{x}$.

$$\text{i.e.} \quad \vec{w} = \underbrace{p(\vec{w},\vec{x})}_{\parallel \vec{x}} + \underbrace{\vec{w} - p(\vec{w},\vec{x})}_{\perp \vec{x}}$$

$$p(\vec{w},\vec{x}) = \left(\vec{w} \circ \frac{\vec{x}}{|\vec{x}|}\right)\frac{\vec{x}}{|\vec{x}|} \qquad \frac{\vec{x}}{|\vec{x}|} = \left(\frac{4}{\sqrt{77}}, \frac{5}{\sqrt{77}}, \frac{6}{\sqrt{77}}\right)$$

$$'' = \frac{32}{\sqrt{77}}\left(\frac{4}{\sqrt{77}}, \frac{5}{\sqrt{77}}, \frac{6}{\sqrt{77}}\right) = \left(\frac{128}{77}, \frac{160}{77}, \frac{192}{77}\right)$$

$$\vec{w} - p(\vec{w},\vec{x}) = \left(1 - \frac{128}{77},\ 2 - \frac{160}{77},\ 3 - \frac{192}{77}\right)$$

This idea is basis for Grahm-Schmidt orthogonalization, a process which creates vectors $\perp$ to each other from a starting set of vectors that are not $\perp$ (orthogonal).

e.g. $\vec{x}$ and $w - p(\vec{w},\vec{x})$ are $\perp$

20 Apr 1989 Linear Algebra – Dot Product (cont.)

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Dot Products: $\vec{x} \circ \vec{y} \equiv x_1 y_1 + \dots + x_N y_N \equiv [x_1, \dots, x_N] \begin{bmatrix} y_1 \\ \vdots \\ y_N \end{bmatrix}$

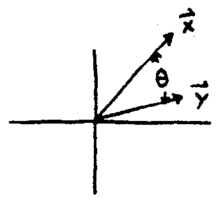
$$|\vec{x}|^2 \equiv x_1^2 + \dots + x_N^2 = \vec{x} \circ \vec{x}$$

geometric view $\vec{x} \circ \vec{y} = |\vec{x}||\vec{y}| \cos\theta$ where $\theta \equiv$ angle between $\vec{x}$ and $\vec{y}$

$|\vec{x}| \equiv$ length of $\vec{x}$

$\vec{x} \circ \vec{y} = 0 \Rightarrow \vec{x} \perp \vec{y}$ or $\vec{x}$ orthogonal to $\vec{y}$

$\cos 90° = 0 \Rightarrow \vec{x} \circ \vec{y} = 0$



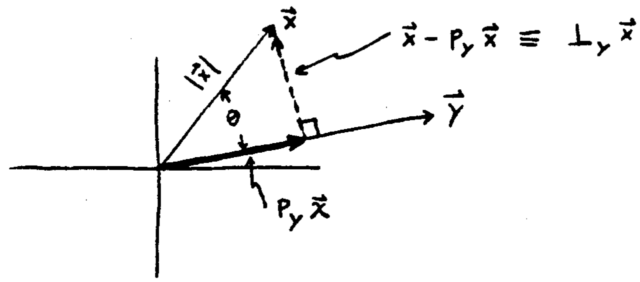
$\vec{x} \circ \vec{y}$ is a scalar

Algebra of $\circ$

$$a(\vec{x} \circ \vec{y}) = a\vec{x} \circ \vec{y} = \vec{x} \circ a\vec{y}$$

$$\vec{x} \circ (\vec{y} + \vec{z}) = \vec{x} \circ \vec{y} + \vec{x} \circ$$

Projections:



We can always write $\vec{x} = P_y\vec{x} + \perp_y \vec{x}$

where $P_y \vec{x} \equiv$ projection of $\vec{x}$ on $\vec{y}$

- Think of this as the amount of $\vec{y}$ in $\vec{x}$, or think of this as the part of $\vec{x}$ that is parallel to $\vec{y}$.

$\perp_y \vec{x} \equiv$ perpendicular component of $\vec{x}$ to $\vec{y}$

- Think of this the thing that makes $\vec{x}$ different from $\vec{y}$, or think of this as the part of $\vec{x}$ that is perpendicular to $\vec{y}$.

As expected $\vec{y} \circ \perp_y \vec{x} = 0$ since vectors are $\perp$

From diagram above: $\left.\begin{array}{l} |P_y\vec{x}| = |\vec{x}| \cos\theta \\ P_y\vec{x} \parallel \vec{y} \quad \text{(parallel to } y) \end{array}\right\}$ $P_y\vec{x} = \left[\vec{x} \circ \dfrac{\vec{y}}{|\vec{y}|}\right] \dfrac{\vec{y}}{|\vec{y}|}$