19 Apr 1990 Linear Algebra — n-Dim Taylor Series

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$$E(\vec{\omega}) = E(\vec{\omega}_0 + \vec{\epsilon}) = \frac{E(\vec{\omega}_0)}{0!} + \frac{\nabla_{\vec{\omega}} E(\vec{\omega})\Big|_{\vec{\omega}=\vec{\omega}_0} \circ \vec{\epsilon}}{1!} + \frac{\vec{\epsilon}^{\,T} H \vec{\epsilon}}{2!} + \ldots$$

where
$$\nabla_{\omega} E(\vec{\omega})\Big|_{\vec{\omega}=\vec{\omega}_0} \equiv \begin{bmatrix} \frac{\partial E(\vec{\omega})}{\partial \omega_1}\Big|_{\vec{\omega}=\vec{\omega}_0} \\ \vdots \\ \frac{\partial E(\vec{\omega})}{\partial \omega_n}\Big|_{\vec{\omega}=\vec{\omega}_0} \end{bmatrix} \equiv \text{gradient}$$

and
$$H \equiv \begin{bmatrix} \frac{\partial^2 E(\vec{\omega})}{\partial \omega_1 \partial \omega_1}\Big|_{\vec{\omega}=\vec{\omega}_0} & \cdots & \frac{\partial^2 E(\vec{\omega})}{\partial \omega_n \partial \omega_1}\Big|_{\vec{\omega}=\vec{\omega}_0} \\ \vdots & & \\ \frac{\partial^2 E(\vec{\omega})}{\partial \omega_1 \partial \omega_n}\Big|_{\vec{\omega}=\vec{\omega}_0} & \cdots & \frac{\partial^2 E(\vec{\omega})}{\partial \omega_n \partial \omega_n}\Big|_{\vec{\omega}=\vec{\omega}_0} \end{bmatrix}$$

$\equiv$ Hessian matrix

ex: $E(\omega_1, \omega_2) = (\omega_1 - 1)^2 + \omega_2^2 + 1$ has min at $\vec{\omega}_0 = (1,0)$

$$\therefore E(1+\epsilon_1, \epsilon_2) = E(\vec{\omega}_0) + \nabla_{\vec{\omega}} E(\vec{\omega})\Big|_{\vec{\omega}_0} \circ \vec{\epsilon} + \vec{\epsilon}^{\,T} H \vec{\epsilon} + \ldots$$

$$= 1 + \begin{bmatrix} 2(\omega_1 - 1)\Big|_{(\omega_1,\omega_2)=(1,0)} \\ 2\omega_2\Big|_{(\omega_1,\omega_2)=(1,0)} \end{bmatrix} \circ \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \end{bmatrix}$$

$$+ \frac{1}{2} [\epsilon_1, \epsilon_2] \begin{bmatrix} 2\Big|_{(\omega_1,\omega_2)=(1,0)} & 0 \\ 0 & 2\Big|_{(\omega_1,\omega_2)=1,0} \end{bmatrix} \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \end{bmatrix} + \ldots$$

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$$E(\vec{\omega}_0 + \vec{\epsilon}) = E(1+\epsilon_1, \epsilon_2) = 1 + \begin{bmatrix} 0 \\ 0 \end{bmatrix} \circ \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \end{bmatrix} + \frac{[\epsilon_1, \epsilon_2]}{2} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \end{bmatrix} + \ldots$$

$$= 1 + 0 + \frac{1}{2} [\epsilon_1, \epsilon_2] \begin{bmatrix} 2\epsilon_1 \\ 2\epsilon_2 \end{bmatrix} + \ldots$$

$$= 1 + \frac{\cancel{2}}{\cancel{2}} \epsilon_1^2 + \frac{\cancel{2}}{\cancel{2}} \epsilon_2^2 + \ldots$$

Check this against 1-Dim slice through $E(\vec{\omega})$:

$$E(1+\epsilon, 0) = (\omega_1 - 1)^2 + 1 = (1 + \epsilon_1 - 1)^2 + 1 = 1 + \epsilon_1^2 \quad \checkmark$$

Note that when we choose $\vec{\omega}_0$ to a minimum of $E(\vec{\omega})$ then the $\nabla E(\vec{\omega})$ is zero. This is because $E(\vec{\omega})$ is flat at a minimum. Near a minimum we always have a quadratic bowl, as we do in the above example.