Ex: Consider a brush DC motor described by the following equations:

$$\frac{di}{dt} = \frac{1}{L}v - \frac{R}{L}i - \frac{K}{L}\omega$$
$$\frac{d\omega}{dt} = \frac{K}{J}i - \frac{1}{J}B\omega$$
$$\frac{d\theta}{dt} = \omega$$

The parameters for the motor are as follows:

L = 5 H

$R = 0\ \Omega$ (the coil has no resistance)

$K = 4$ Nm/A

$J = 0.8$ kg m^2

$B = 1.6$ Nms

Suppose the following form of speed control is chosen for the motor:

$$v = K_P(\omega_{ref} - \omega) + K_I \int_0^t (\omega_{ref} - \omega)dt$$

a) Draw a block diagram showing the s-domain model of the control system and motor.

b) Write an expression for the speed control transfer function,

$$H(s) = \frac{\Omega(s)}{\Omega_{ref}(s)}$$

where

$$\Omega(s) \equiv \text{Laplace Transform of } \omega(t)$$
$$\Omega_{ref}(s) \equiv \text{Laplace Transform of } \omega_{ref}(t)$$

c) Draw a pole zero plot for H(s) when K_P is very small and $K_I/K_P = 2$.

d) For K_P a very small positive number, we may approximate H(s) in terms of partial fractions as a pole term for $s = 0$ plus other pole terms:

$$H(s) \approx \frac{A}{s} + \text{other pole terms}$$

Write A as $A = K_P x$ and find a good estimate for the magnitude of x.

sol'n: a) Laplace Transform the motor eq'ns:

$$sI(s) = \frac{1}{L}V(s) - \overbrace{\cancel{\frac{R}{L}}}^{0 \text{ since } R=0} I(s) - \frac{K}{L}\Omega(s)$$

$$s\Omega(s) = \frac{K}{J}I(s) - \frac{1}{J}B\,\Omega(s)$$

$$s\Theta(s) = \Omega(s)$$

Solve 1st eq'n for $I(s)$ and substitute into 2nd eq'n to get eq'n for $\Omega(s)$ in terms of $V(s)$, (since we have V control).

$$s\Omega(s) = \frac{K}{sLJ}V(s) - \frac{K^2}{sLJ}\Omega(s) - \frac{B}{J}\Omega(s)$$

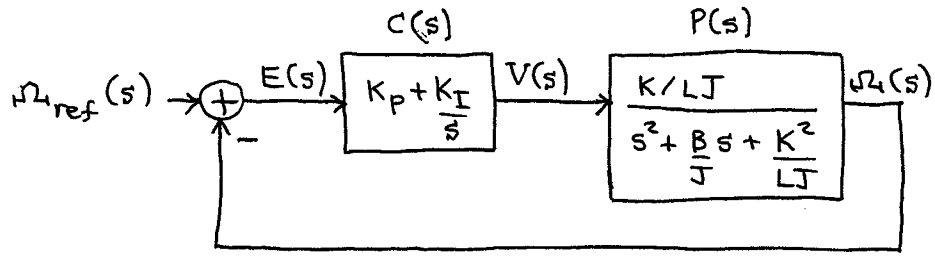
$$\text{or } \left(s^2 + \frac{B}{J}s + \frac{K^2}{LJ}\right)\Omega(s) = \frac{K}{LJ}V(s)$$

$$\therefore \frac{\Omega(s)}{V(s)} = \frac{K/LJ}{s^2 + \frac{B}{J}s + \frac{K^2}{LJ}} \equiv P(s) \quad \text{(motor)}$$

Laplace Transform the control strategy:

$$V(s) = K_P\left[\Omega_{ref}(s) - \Omega(s)\right] + \left[\frac{K_I}{s}\Omega_{ref}(s) - \Omega(s)\right]$$

$$\text{or } V(s) = \underbrace{\left(K_P + \frac{K_I}{s}\right)}_{C(s)}\underbrace{\left(\Omega_{ref}(s) - \Omega(s)\right)}_{E(s)}$$



b) From eq'ns of system or from block diagram

$H(s) \equiv \dfrac{\Omega(s)}{\Omega_{ref}(s)}$ is found by starting at $\Omega(s)$

and working back thru loop.

$$\Omega(s) = C(s)P(s)\left[\Omega_{ref}(s) - \Omega(s)\right]$$

$$\Omega(s)\left[1 + C(s)P(s)\right] = C(s)P(s)\Omega_{ref}(s)$$

$$\therefore \quad H(s) \equiv \frac{\Omega(s)}{\Omega_{ref}(s)} = \frac{C(s)P(s)}{1 + C(s)P(s)}$$

Now reduce $H(s)$ to a ratio of polynomials in s.

$$H(s) = \frac{\frac{K}{LJ}\left(K_p + \frac{K_I}{s}\right) \Big/ \left(s^2 + \frac{B}{J}s + \frac{K^2}{LJ}\right)}{1 + \frac{K}{LJ}\left(K_p + \frac{K_I}{s}\right) \Big/ \left(s^2 + \frac{B}{J}s + \frac{K^2}{LJ}\right)}$$

$$H(s) = \frac{\frac{K_pK}{LJ}\left(s + K_I/K_p\right) \Big/ \left(s^2 + \frac{B}{J}s + \frac{K^2}{LJ}\right)}{s + \frac{K_pK}{LJ}\left(s + K_I/K_p\right) \Big/ \left(s^2 + \frac{B}{J}s + \frac{K^2}{LJ}\right)}$$

$$H(s) = \frac{\frac{K_P K}{LJ}\left(s + K_I/K_P\right)}{s\left(s^2 + \frac{B}{J}s + \frac{K^2}{LJ}\right) + K_P \frac{K}{LJ}\left(s + K_I/K_P\right)}$$

c) Calculate numerical values of coefficients:

$$\frac{K}{LJ} = \frac{4\ \text{Nm/A}}{5\ \text{H}\ 0.8\ \text{kg m}^2} = 1\ \frac{\cancel{\text{kg}}\,\cancel{\text{m}^2}/\cancel{\text{A}}\,\text{s}^2}{\text{Vs}/\cancel{\text{A}} \cdot \cancel{\text{kg}}\,\cancel{\text{m}^2}}$$

$$\frac{K}{LJ} = 1\ (1/\text{Vs}^3)$$

$$\frac{B}{J} = \frac{1.6\ \text{Nms}}{0.8\ \text{kg m}^2} = 2\ \frac{\cancel{\text{kg}}\,\cancel{\text{m}^2}/\text{s}}{\cancel{\text{kg}}\,\cancel{\text{m}^2}}$$

$$\frac{B}{J} = 2\ (1/\text{s})$$

$$\frac{K^2}{LJ} = K \cdot \frac{K}{LJ} = 4\ \text{Nm/A} \cdot 1\ \ 1/\text{Vs}^3$$

$$\frac{K^2}{LJ} = 4\ (1/\text{s}^2)$$

For K_P small, the poles obtained by setting denominator of $H(s)$ equal to zero are the solutions of following eq'n:

$$s\left(s^2 + \frac{B}{J}s + \frac{K^2}{LJ}\right) = 0$$

or $s\left(s^2 + 2s + 4\right) = 0$

poles are $s = 0$, $s = -1 \pm j\sqrt{3}$

We calculate zeros by setting numerator of $H(s)$ equal to zero.

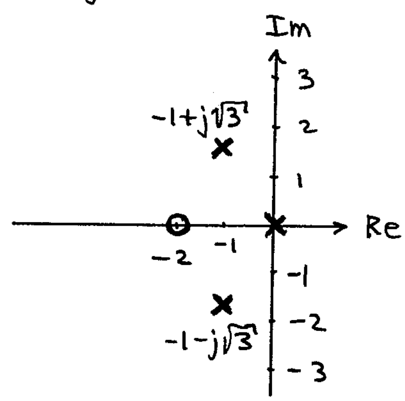
$$\frac{K_p K}{LJ}\left(s + K_I / K_p\right) = 0$$

or $\left(s + K_I / K_p\right) = 0$

or $s + 2 = 0$

zero at $s = -2$

pole-zero diagram:



d) We have $H(s) = \frac{A}{s} + \ldots$ in partial fraction expansion. Since $A = sH(s)\Big|_{s=0}$,

we can find A from the pole-zero diagram by removing the pole at $s=0$ (which is equivalent to multiplying $H(s)$ by s) and evaluating the function $sH(s)$ for $s=0$. In particular, we have $|A| = \frac{K_p K}{LJ} \prod_{\text{zeros}} \text{dist zeros to } s \Big/ \prod_{\text{poles}} \text{dist poles to } s.$

Thus $|A| = \frac{K_p K}{LJ} \; |-2-0| \Big/ \left|-1+j\sqrt{3}-0\right| \left|-1-j\sqrt{3}-0\right|$

$$= K_p \cdot 1 \left(1/V s^3\right) \frac{2}{\sqrt{1^2+\sqrt{3}^2}\;\sqrt{1^2+\sqrt{3}^2}}$$

$$|A| = K_p \cdot 1/V s^3 \; \frac{2}{2 \cdot 2} = \frac{K_p}{2}$$