Ex: Write the equations below describing a two-phase induction motor, [1], (with no friction and number of pole pairs, n_p, equal to one) in terms of rotation matrices.

$$\frac{d}{dt}\begin{bmatrix} i_{SA} \\ i_{SB} \\ i_{RX} \\ i_{RY} \end{bmatrix} = \frac{1}{L_S L_R - M^2}\begin{bmatrix} L_R & 0 & -M\cos(\theta) & M\sin(\theta) \\ 0 & L_R & -M\sin(\theta) & -M\cos(\theta) \\ -M\cos(\theta) & -M\sin(\theta) & L_S & 0 \\ M\sin(\theta) & -M\cos(\theta) & 0 & L_S \end{bmatrix}$$

$$\cdot\begin{bmatrix} v_{SA} - R_S i_{SA} + M i_{RX}\omega\sin(\theta) + M i_{RY}\omega\cos(\theta) \\ v_{SB} - R_S i_{SB} - M i_{RX}\omega\cos(\theta) + M i_{RY}\omega\sin(\theta) \\ -R_R i_{RX} + M i_{SA}\omega\sin(\theta) - M i_{SB}\omega\cos(\theta) \\ -R_R i_{RY} + M i_{SA}\omega\cos(\theta) + M i_{SB}\omega\sin(\theta) \end{bmatrix}$$

$$\frac{d\omega}{dt} = \frac{M}{J}(-i_{SA} i_{RX}\sin(\theta) - i_{SA} i_{RY}\cos(\theta) + i_{SB} i_{RX}\cos(\theta) - i_{SB} i_{RY}\sin(\theta))$$

That is, fill in the empty brackets in the equations below using only 2×2 matrices or products of 2×2 matrices written symbolically as U_θ or I multiplied by constants. You may also use transposes or U matrices for specific angles.

$$\frac{d}{dt}\begin{bmatrix} \vec{i}_S \\ \vec{i}_R \end{bmatrix} = \frac{1}{L_S L_R - M^2}\begin{bmatrix} L_R[\ ] & -M[\ ] \\ -M[\ ] & L_S[\ ] \end{bmatrix}\left(\begin{bmatrix} [\ ] & [\mathbf{0}] \\ [\mathbf{0}] & [\mathbf{0}] \end{bmatrix}\begin{bmatrix} \vec{v}_S \\ \vec{0} \end{bmatrix} + \begin{bmatrix} R_S[\ ] & M\omega[\ ] \\ M\omega[\ ] & R_R[\ ] \end{bmatrix}\begin{bmatrix} \vec{i}_S \\ \vec{i}_R \end{bmatrix}\right)$$

$$\frac{d}{dt}\omega = \frac{M}{J}\vec{i}_S^{\,T}[\ ]\vec{i}_R$$

Use the following definitions in your solution:

$$\vec{i}_S \equiv \begin{bmatrix} i_{SA} \\ i_{SB} \end{bmatrix} \qquad \vec{i}_R \equiv \begin{bmatrix} i_{RX} \\ i_{RY} \end{bmatrix} \qquad \vec{v}_S \equiv \begin{bmatrix} v_{SA} \\ v_{SB} \end{bmatrix}$$

$$I \equiv \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad U_\theta \equiv \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

SOL'N: Consider a 90° rotation matrix and combinations with rotations by θ:

$$U_{90°} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$U_{90°}U_{-\theta} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} \cos(-\theta) & \sin(-\theta) \\ -\sin(-\theta) & \cos(-\theta) \end{bmatrix} = \begin{bmatrix} \sin\theta & \cos\theta \\ -\cos\theta & \sin\theta \end{bmatrix}$$

$$U_{\theta}U_{-90°} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \sin\theta & -\cos\theta \\ \cos\theta & \sin\theta \end{bmatrix}$$

$$U_{-\theta}U_{-90°} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -\sin\theta & -\cos\theta \\ \cos\theta & -\sin\theta \end{bmatrix}$$

We also use the identity that the transpose of the U matrix for a rotation by θ is equal to the U matrix for a rotation by –θ.

Using the above information, we obtain the following expression for the induction motor equations:

$$\frac{d}{dt}\begin{bmatrix} \vec{i}_S \\ \vec{i}_R \end{bmatrix} = \frac{1}{L_S L_R - M^2}\begin{bmatrix} L_R[I] & -M[U_{-\theta}] \\ -M[U_{\theta}] & L_S[I] \end{bmatrix} \cdot \left(\begin{bmatrix} [I] & [\mathbf{0}] \\ [\mathbf{0}] & [\mathbf{0}] \end{bmatrix}\begin{bmatrix} \vec{v}_S \\ \vec{0} \end{bmatrix} + \begin{bmatrix} R_S[-I] & M\omega[U_{-\theta}U_{90°}] \\ M\omega[U_{\theta}U_{-90°}] & R_R[-I] \end{bmatrix}\begin{bmatrix} \vec{i}_S \\ \vec{i}_R \end{bmatrix} \right)$$

$$\frac{d}{dt}\omega = \frac{M}{J}\vec{i}_S^{\,T}[U_{-\theta}U_{-90°}]\vec{i}_R$$

REF: [1] Marc Bodson, "Control of Electric Motors," 2004, University of Utah ECE Dept., eqn 4.27 p. 130.