ex: Consider a permanent magnet motor

$$\frac{di}{dt} = \frac{1}{L}v - \frac{R}{L}i + \frac{\Psi_0}{L}\omega\sin(\theta)$$

$$\frac{d\omega}{dt} = -\frac{\Psi_0}{J}\sin(\theta)\,i - \frac{\tau_L}{J}$$

$$\frac{d\theta}{dt} = \omega$$

This system of eq'ns is written <u>autonomous form</u>:

$$\dot{x}_1 = f_1(x_1, x_2, x_3)$$

$$\dot{x}_2 = f_2(x_1, x_2, x_3)$$

$$\dot{x}_3 = f_3(x_1, x_2, x_3)$$

where $x_1 \equiv i$, $x_2 \equiv \omega$, $x_3 \equiv \theta$

Note that we have 1^{st} derivatives of state vars on the left.
Note that we have only state vars (no derivatives or other variables) and constants on the right.
Note that the third eq'n establishes the relationship that $\theta = \int \omega\, dt$.

ex: We analyze the autonomous system of eq'ns by linearizing them. We do this because we may not be able to solve the original eq'ns.

Although we can linearize our eq'ns around any point $\vec{x}_0 \equiv (i_0, \omega_0, \theta_0)$, we are often interested in linearizing around fixed points (where the system is likely to come to rest).

The fixed pts depend on values of constants.

We will find fixed pts symbolically.

The fixed pts occur where all the derivatives equal zero:

$$\frac{di}{dt} = 0 = \frac{1}{L}v - \frac{R}{L}i + \frac{\psi_0}{L}\omega\sin(\theta)$$

$$\frac{d\omega}{dt} = 0 = -\frac{\psi_0}{J}\sin(\theta)\,i - \frac{\tau_L}{J}$$

$$\frac{d\theta}{dt} = 0 = \omega$$

Now we find the possible values for state vars.

From the 3rd eq'n, $\omega = 0$. Substituting this into the 1st eq'n yields $0 = \frac{1}{L}v - \frac{R}{L}i$.

Thus, $i = \frac{v}{R}$, (Ohm's Law).

The 2nd eq'n now becomes $0 = -\frac{\psi_0}{J}\sin(\theta)\frac{v}{R} - \frac{\tau_L}{J}$.

Thus, $\sin(\theta) = -\frac{\tau_L}{J}\frac{J}{\psi_0}\frac{R}{v} = \frac{-\tau_L R}{\psi_0 v}$

$$\text{or}\quad \theta = \sin^{-1}\left(\frac{-\tau_L R}{\psi_0 v}\right)$$

For the sake of example, suppose $\frac{-\tau_L R}{\psi_0 v} = \frac{-1}{\sqrt{2}}$.

Then $\theta = -45°$ or $-135°$, i.e., $-\frac{\pi}{4}$ and $-\frac{3\pi}{4}$.

In this case, our fixed pts would be

$$(i_0, \omega_0, \theta_0) = \left(\frac{v}{R}, 0, -\frac{\pi}{4}\right)$$

and $(i_0, \omega_0, \theta_0) = \left(\frac{v}{R}, 0, -\frac{3\pi}{4}\right)$.

ex: Now consider linearization around the fixed pt $(i_0, \omega_0, \theta_0) = \left(\frac{v}{R}, 0, -\frac{\pi}{4}\right)$.

We use a matrix Taylor series (for the right side of the autonomous eqns) expanded around the fixed point.

First, we write state vars in terms of perturbations:

$$i = i_0 + \epsilon_i, \quad \omega = \omega_0 + \epsilon_\omega, \quad \theta = \theta_0 + \epsilon_\theta$$

We assume the perturbations are small and drop Taylor series terms that are quadratic or higher order in ϵ_i, ϵ_ω, or ϵ_θ.

What we now have is: constant term + linear term

$$\begin{array}{l} \frac{d(i_0+\epsilon_i)}{dt} \doteq \\ \\ \frac{d(\omega_0+\epsilon_\omega)}{dt} \doteq \\ \\ \frac{d(\theta_0+\epsilon_\theta)}{dt} \doteq \end{array} \underbrace{\begin{bmatrix} \frac{1}{L}v - \frac{R}{L}i_0 + \frac{\Psi_0}{L}\omega_0 \sin\theta_0 \\ \\ -\frac{\Psi_0}{J}\sin(\theta_0) i_0 - \frac{\tau_L}{J} \\ \\ \omega_0 \end{bmatrix}}_{\text{eqns evaluated at } (i_0,\omega_0,\theta_0) = \vec{0} \text{ since fixed pt}} + Df\Big|_{i_0,\omega_0,\theta_0} \cdot \begin{bmatrix} \epsilon_i \\ \\ \epsilon_\omega \\ \\ \epsilon_\theta \end{bmatrix}$$

where Df is the Jacobian matrix:

$$Df \equiv \begin{bmatrix} \frac{\partial f_1(i,\omega,\theta)}{\partial i} & \frac{\partial f_1(i,\omega,\theta)}{\partial \omega} & \frac{\partial f_1(i,\omega,\theta)}{\partial \theta} \\ \frac{\partial f_2(i,\omega,\theta)}{\partial i} & \frac{\partial f_2(i,\omega,\theta)}{\partial \omega} & \frac{\partial f_2(i,\omega,\theta)}{\partial \theta} \\ \frac{\partial f_3(i,\omega,\theta)}{\partial i} & \frac{\partial f_3(i,\omega,\theta)}{\partial \omega} & \frac{\partial f_3(i,\omega,\theta)}{\partial \theta} \end{bmatrix}$$

where $f_1(i,\omega,\theta)$, $f_2(i,\omega,\theta)$, $f_3(i,\omega,\theta)$ are

Now suppose we have the following numerical values: $\frac{R}{L} = 3$, $\frac{\Psi_0 v}{JR\sqrt{2}} = \frac{6}{100}$, $\frac{\Psi_0^2}{2LJ} = \frac{203}{100}$

$\frac{\Psi_0}{J\sqrt{2}} = 1$, $\frac{\Psi_0}{L\sqrt{2}} = 203/100$

Then $\dot{\vec{\epsilon}} = A\vec{\epsilon}$ where

$$A = \begin{bmatrix} -\frac{R}{L} & -\frac{\Psi_0}{L\sqrt{2}} & 0 \\ \frac{\Psi_0}{J\sqrt{2}} & 0 & -\frac{\Psi_0 v}{JR\sqrt{2}} \\ 0 & 1 & 0 \end{bmatrix},$$

and we find eigenvalues by solving $|\lambda I - A| = 0$ where $|\ | \equiv$ determinant.

At this point we observe that the eigenvalues are the same as the roots of the characteristic eq'n for the system in the Laplace domain.

In other words, the eigenvalues are the poles of the system.

We see this when we take the Laplace transform of the system, $\mathcal{L}\{\dot{\vec{\epsilon}} = A\vec{\epsilon}\}$, to get

$$s\vec{E} = A\vec{E} \qquad \text{where } \vec{E} \equiv \vec{E}(s) \equiv \mathcal{L}\{\vec{\epsilon}(t)\}$$

$$\text{or} \quad sI\vec{E} = A\vec{E}$$

$$\text{or} \quad (sI - A)\vec{E} = 0$$

This is the same as the eq'n for eigenvalues.

For our example we want to solve $|sI - A| = 0$ or

$$\begin{vmatrix} s + \frac{R}{L} & \frac{\Psi_0}{L\sqrt{2}} & 0 \\ -\frac{\Psi_0}{J\sqrt{2}} & s & \frac{\Psi_0 v}{JR\sqrt{2}} \\ 0 & -1 & s \end{vmatrix} = \left(s + \frac{R}{L}\right)\left(s^2 + \frac{\Psi_0 v}{JR\sqrt{2}}\right) - \frac{\Psi_0}{L\sqrt{2}}\left(-\frac{\Psi_0}{J\sqrt{2}}\right)s = 0$$

or $(s+3)\left(s^2+\frac{6}{100}\right)+\frac{203}{100}s=0$

or $s^3+3s^2+\left(\frac{6}{100}+\frac{203}{100}\right)s+3\cdot\frac{6}{100}=0$

or $(s+2)\left(s+\frac{9}{10}\right)\left(s+\frac{1}{10}\right)=0$

Thus, our eigenvalues (or poles) are

$\lambda_1=p_1=-2 \quad \lambda_2=p_2=-\frac{9}{10} \quad \lambda_3=p_3=-\frac{1}{10}$.

Now find eigenvectors:

$p_1=-2$
use for s

$$\begin{vmatrix} 1 & \frac{203}{100} & 0 \\ -1 & -2 & \frac{6}{100} \\ 0 & -1 & -2 \end{vmatrix}\begin{bmatrix}\epsilon_1\\\epsilon_2\\\epsilon_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

From 3rd line $-\epsilon_2-2\epsilon_3=0 \Rightarrow \epsilon_3=\frac{-1}{2}\epsilon_2$
From 1st line $\epsilon_1+\epsilon_2\left(\frac{203}{100}\right)=0 \Rightarrow \epsilon_1=-\frac{203}{100}\epsilon_2$

If $\epsilon_2=1$, then $\epsilon_3=-\frac{1}{2}$, $\epsilon_1=-\frac{203}{100}$.

This sol'n satisfies 2nd line, too. ✓

$p_2=\frac{-9}{10}$

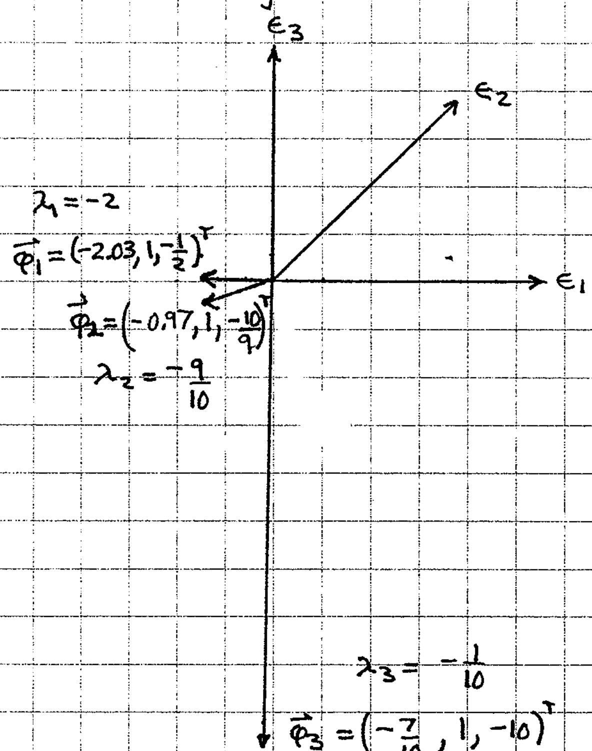
$$\begin{vmatrix} \frac{21}{10} & \frac{203}{100} & 0 \\ -1 & -\frac{9}{10} & \frac{6}{100} \\ 0 & -1 & -\frac{9}{10} \end{vmatrix}\begin{bmatrix}\epsilon_1\\\epsilon_2\\\epsilon_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

If $\epsilon_2=1$, then $\epsilon_1=\frac{-10}{21}\cdot\frac{203}{100}=-\frac{203}{210}$, $\epsilon_3=-\frac{10}{9}$

$p_1=\frac{-1}{10}$

If $\epsilon_2=1$, then $\epsilon_1=-\frac{10}{29}\frac{203}{100}=-\frac{7}{10}$, $\epsilon_3=-10$

Plot of eigenvectors:



Note: These eigenvectors are not normalized to have length = 1.

The time-domain solution of our system will be of form

$$\begin{bmatrix} i(t) \\ \omega(t) \\ \theta(t) \end{bmatrix} = a_1 e^{\lambda_1 t} \vec{\varphi}_1 + a_2 e^{\lambda_2 t} \vec{\varphi}_2 + a_3 e^{\lambda_3 t} \vec{\varphi}_3$$

where a_1, a_2, a_3 are constants determined by init cond.

Note: $\vec{\varphi}_1, \vec{\varphi}_2, \vec{\varphi}_3$ form a basis (of vectors) in terms of which we write our sol'n.
The eigvecs, $\vec{\varphi}_i$, are the directions sol'ns follow.
The λ_i's (eigvals) determine how fast the sol'n moves in the directions of $\vec{\varphi}_i$'s.
λ_i's = poles in left-half plane = stable sys.