EX: Five fair 6-sided dice are rolled at once. Then all five dice are rolled again. This repeats until the five dice have been rolled ten times. Find the probability that a full house occurs on exactly four of the rolls. A full house is when exactly three dice show the same number, (i.e., three of a kind), and the remaining two dice show the same number, (i.e., a pair). Note that the number showing on the pair must be different from the number showing on the three of a kind, as otherwise we would have five-of-a-kind.

SOL'N: From an earlier example, we know that $P(\text{full house}) = \dfrac{300}{6^5}$ for one roll.

Since the outcome for each roll is independent of the outcomes for other rolls, the probability of a certain sequence of outcomes on successive rolls is the product of the probabilities of the outcomes for each roll. The probability of four full houses followed by six non-full houses is

$$\begin{aligned} &P(\text{4 full houses followed by 6 non-full houses}) \\ &\quad = P(\text{full house})^4\, P(\text{non-full house})^6 \\ &\quad = \left(\frac{300}{6^5}\right)^4 \cdot \left(1 - \frac{300}{6^5}\right)^6 \approx 2.215 \cdot 10^{-6} \ \cdot\ 0.7897 \approx 1.75 \cdot 10^{-6} \end{aligned}$$

Since multiplication is commutative, any sequence of ten rolls in which four rolls are full houses and six rolls are non-full houses will yield the same product. This means that the total probability of exactly four full houses in ten rolls will be the number of sequences of rolls with exactly four full houses multiplied by $1.75 \cdot 10^{-6}$.

When we speak of a sequence of rolls, we are categorizing each roll as a full house or a non-full house. We may think of a full house as a "1" and a non-full house as a "0". To specify a sequence of ten rolls, we write a binary number of ten digits. The binary number 1111000000, for example, represents four full houses followed by four non-full houses. The list of all possible sequences turns out to be the same as the list of all possible 10-digit binary numbers:

0000000000

0000000001

0000000010

0000000011

0000000100

⋮

1111111111

The number of sequences with exactly four full houses is the number of 10-digit binary numbers with exactly four 1's. There are $2^{10} = 1024$ of these 10-digit binary numbers. Listing them all and counting how many have four 1's would be tedious. Instead, we observe that this count will be the same as the number of ways of picking 4 out of 10 objects. Since we are only interested in the *set* of four digits that are 1's, and not the time-order in which we choose the four digits that are 1's, we use a combinatoric coefficient.

$$\text{\# 10-digit binary \#'s with exactly four 1's} = {}_{10}C_4 = \frac{10!}{6!4!} = \frac{10 \cdot 9 \cdot 8 \cdot 7}{4 \cdot 3 \cdot 2 \cdot 1} = 210$$

Taking the product of this number and the probability of a particular sequence with exactly four full houses gives the probability of four full houses in ten rolls.

$$P(\text{exactly 4 full houses in 10 rolls}) = 210 \cdot 1.75 \cdot 10^{-6} \approx 3.66 \cdot 10^{-6}$$

This is about 1 in 2721. Since one trial involves 10 rolls, this means this would happen, on average, only once in 27,210 rolls.