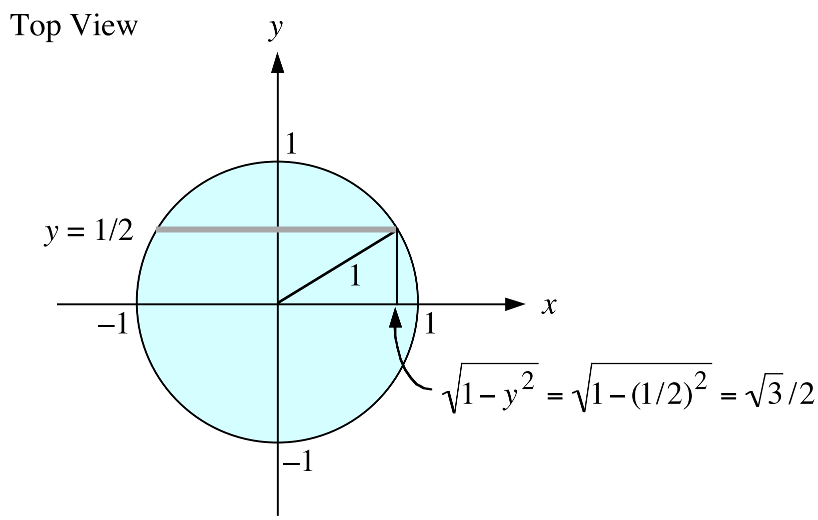
EX: A joint probability density function is defined as follows:

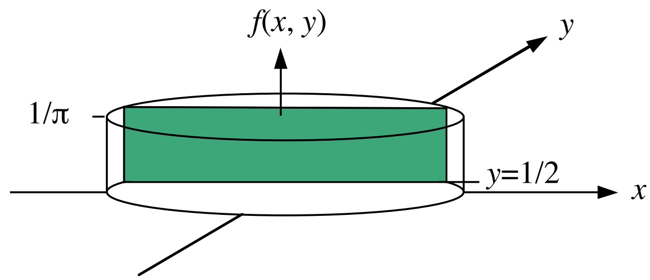
$$f(x,y) = \begin{cases} \frac{1}{\pi} & x^2 + y^2 \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Find the conditional probability $f(x \mid y = \tfrac{1}{2})$.

SOL'N: The region, $x^2 + y^2 \le 1$, on which $f(x, y) \ne 0$ is called the support of $f(x, y)$. It is a circle of radius one, centered on the origin, as shown below. The diagram also shows the horizontal segment that is the support for the cross-section that forms the basis for $f(x \mid y = \tfrac{1}{2})$.



The diagram shows that the cross-section extends from $-\sqrt{3}/2$ to $\sqrt{3}/2$. The conditional probability, $f(x \mid y = \tfrac{1}{2})$, is a scaled version of the cross section of $f(x, y)$ at $y = 1/2$. The illustration, below, shows the 3-dimensional shape of $f(x, y)$ and the cross section in the x direction at $y = 1/2$.



The probability density function that is $f(x \mid y = \frac{1}{2})$ is the above cross-section scaled vertically to have an area equal to one. Since the cross-section is rectangular, this means the height will be scaled up to a value of 1/width where the width is $2\sqrt{3}/2 = \sqrt{3}$:

$$f(x \mid y = \frac{1}{2}) = \begin{cases} 1/\sqrt{3} & -\sqrt{3}/2 \le x \le \sqrt{3}/2 \\ 0 & \text{otherwise} \end{cases}$$

Taking a more strictly mathematical approach, we would integrate to find the area of the cross section and divide the cross-section by the result:

$$f(x \mid y = \frac{1}{2}) = \begin{cases} \dfrac{f(x, y = \frac{1}{2})}{\int_{-\sqrt{3}/2}^{\sqrt{3}/2} f(x, y = \frac{1}{2})dx} & -\sqrt{3}/2 \le x \le \sqrt{3}/2 \\ 0 & \text{otherwise} \end{cases}$$

or

$$f(x \mid y = \frac{1}{2}) = \begin{cases} \dfrac{\frac{1}{\pi}}{\int_{-\sqrt{3}/2}^{\sqrt{3}/2} \frac{1}{\pi} dx} & -\sqrt{3}/2 \le x \le \sqrt{3}/2 \\ 0 & \text{otherwise} \end{cases}$$

The integral is the difference of the limits multiplied by 1/π.

$$f(x \mid y = \frac{1}{2}) = \begin{cases} \dfrac{\frac{1}{\pi}}{2\frac{\sqrt{3}}{2}\frac{1}{\pi}} & -\sqrt{3}/2 \le x \le \sqrt{3}/2 \\ 0 & \text{otherwise} \end{cases}$$

Canceling out terms yields the same result as before:

$$f(x \mid y = \frac{1}{2}) = \begin{cases} 1/\sqrt{3} & -\sqrt{3}/2 \le x \le \sqrt{3}/2 \\ 0 & \text{otherwise} \end{cases}$$