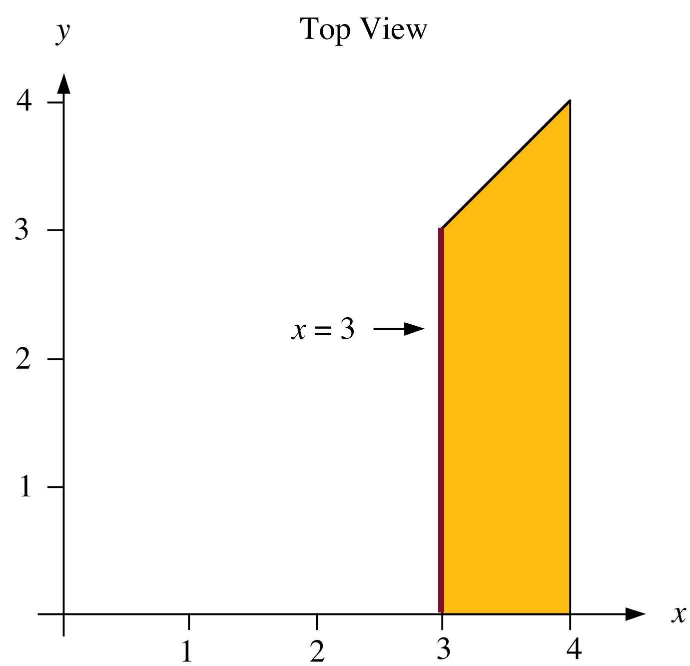
EX: A joint probability density function is defined as follows:

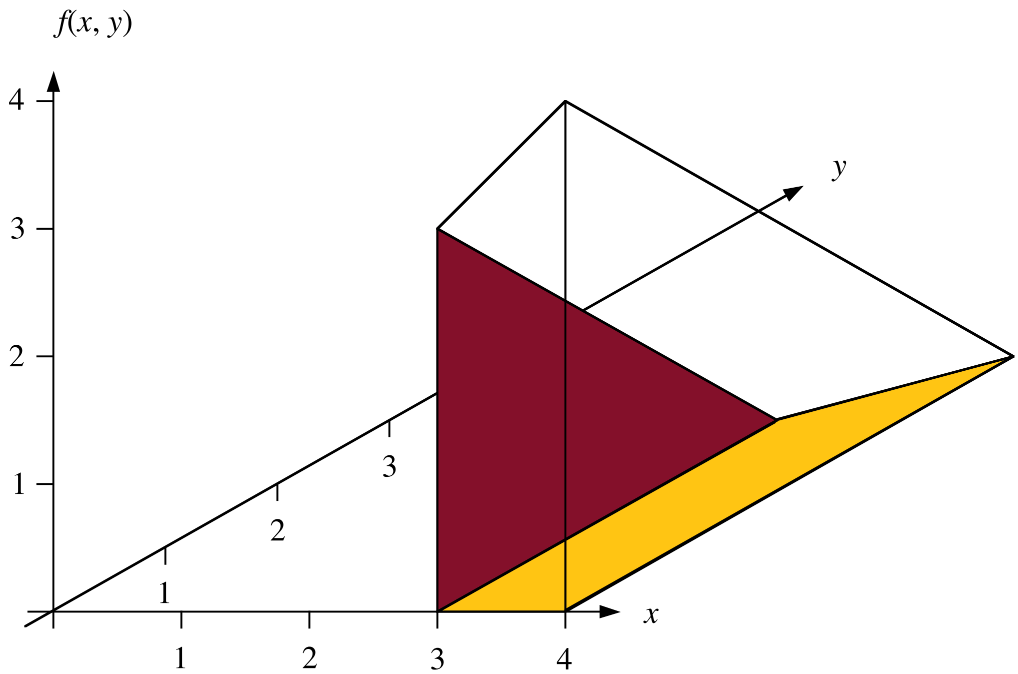
$$f(x,y) = \begin{cases} x - y & 3 \le x \le 4 \text{ and } 0 \le y \le x \\ \\ 0 & \text{otherwise} \end{cases}$$

Find the conditional probability $f(y \mid x = 3)$.

SOL'N: The region, $3 \le y \le 4$ and $0 \le y \le x$, on which $f(x, y) \ne 0$ is the support of $f(x, y)$. It is a trapezoid, as shown below. The diagram also shows the vertical segment for $x = 3$.



The illustration, below, shows the 3-dimensional shape of $f(x, y)$. The figure also shows cross-sections at $x = 3$. The density function for $f(y \mid x = 3)$ is equal to the area of the cross-section of $f(x, y)$ at $x = 3$ scaled vertically to have an area equal to one.



Mathematically, we are using the values of $f(x = 3, y)$.

$$f(x=3,y)=\begin{cases}3-y & 0\le y\le 3\\ \\ 0 & \text{otherwise}\end{cases}$$

The vertical scaling is taken care of by the integral in the denominator of the equation for conditional probability.

$$f(y\mid x=3)=\frac{f(x=3,y)}{\int_{y=0}^{y=3} f(x=3,y)\,dy}$$

or

$$f(y\mid x=3)=\frac{3-y}{\int_{y=0}^{y=3}(3-y)\,dy}$$

The integral in the denominator is the area of the triangular cross-section at $x = 3$. We may compute it as one-half base times height or by integrating:

$$\int_{y=0}^{y=3} (3-y)\,dy = \left(3y - \frac{y^2}{2}\right)\Bigg|_{y=0}^{y=3} = 9 - \frac{9}{2} = \frac{9}{2}$$

Substituting this value into our earlier equation for $f(y \mid x = 3)$ gives our answer:

$$f(y \mid x = 3) = \frac{3-y}{9/2} = \frac{2}{9}(3-y)$$