EX: For the Spy Game (see Conceptual Tools: Probability: Conditional Probability: Discrete random variables: spy game), we calculate the following probabilities:

1) P(player = yellow | public die = 2 & result = hit)

2) P(player = orange | public die = 2 & result = miss)

3) P(player = blue | 1st public die = 2, result = miss, 2nd public die =3, result = miss)

SOL'N: a) From the game board, which has one spots out of six for yellow on the first row (i.e., for public die = 2), we might guess that the probability that the player color is yellow is 1/6, or we might think yellow should be more likely, since we now know the player's color cannot be red or purple.

For the formal calculation, we want a diagram showing possible outcomes. Consider first the following characteristics of a player's turn:

- public die number
- private die number
- player color

These characteristics are independent. The value of each of them is not influenced by the values of the other two. Note that we are not talking about the player color after we know whether each turn is a hit or miss. The hits and misses have yet to make an appearance. Note also that we may add the die numbers from as many turns as desired to this list of independent characteristics.

The value of each characteristic is equally likely to be one of six values. Using independence, we have $6^3 = 216$ equally likely possible outcomes:

$$P(\text{public die} = n, \text{private die} = m, \text{player color} = x) = \frac{1}{6}\cdot\frac{1}{6}\cdot\frac{1}{6} = \frac{1}{216}$$

for viable values of n, m, and x.

Fig. 1 below illustrates the probabilities. Each dot represents an outcome with probability 1/216. Since we need three axes, we replicate the 2-axis diagram for each color.

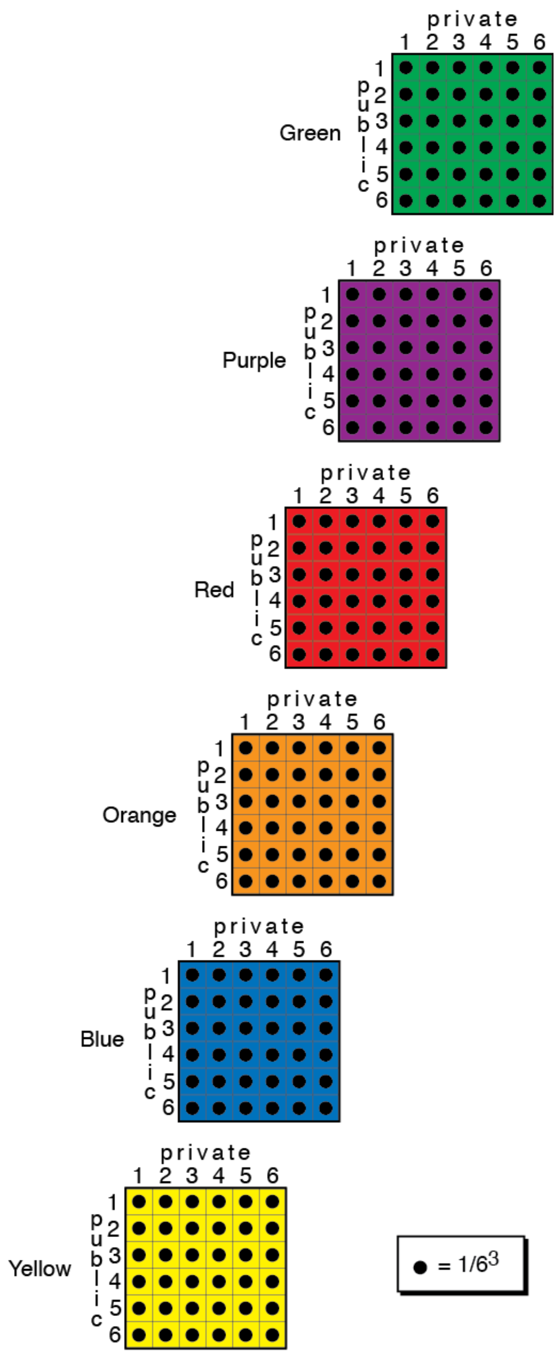

Fig. 1. Probability of public die, private die, and player color in spy game.

Now we add the hit or miss result. The hit or miss splits the outcomes into two events deterministically. Thus, we add a fourth axis to our diagram.

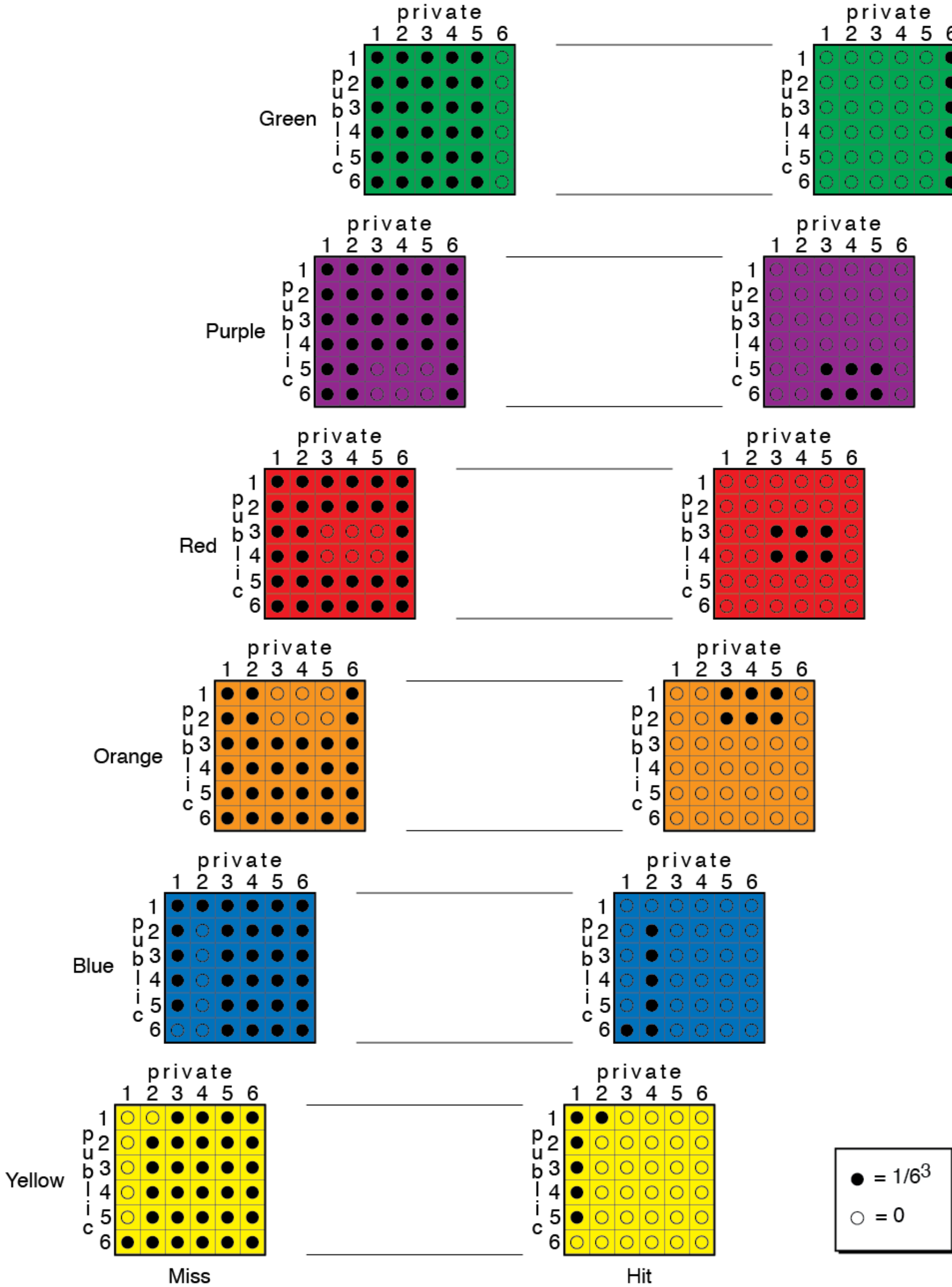

Fig. 2. Probability of public die, private die, player color, and hit or miss in spy game.

Adding the restriction that public die = 2, we have the diagram in Fig. 3.

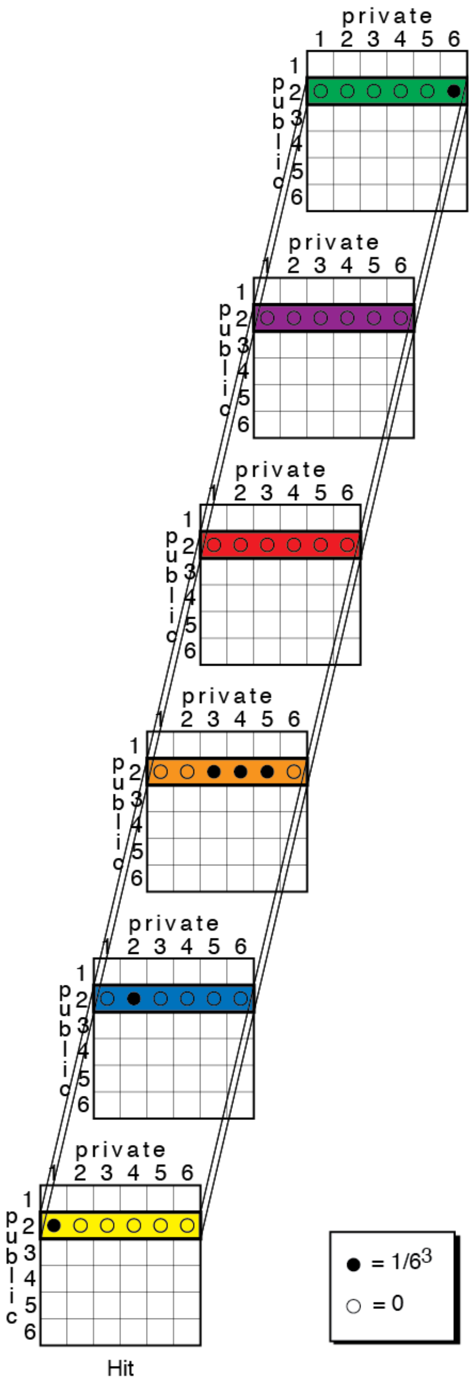

Fig. 3. Probability of public die = 2, private die, player color, and hit in spy game.

Now we perform our calculations.

$$P(\text{player} = \text{yellow} \mid \text{public die} = 2, \text{hit})$$
$$= \frac{P(\text{player} = \text{yellow}, \text{public die} = 2, \text{hit})}{P(\text{player} = \text{any color}, \text{public die} = 2, \text{hit})}$$

Our notation is a bit misleading. We should write the following to include the unknown private die number:

$$P(\text{player} = \text{yellow, private die} = \text{any} \mid \text{public die} = 2, \text{hit})$$
$$= \frac{P(\text{player} = \text{yellow, private die} = \text{any, public die} = 2, \text{hit})}{P(\text{player} = \text{any color, private die} = \text{any, public die} = 2, \text{hit})}$$

We count the dots that meet the given specifications, each of which has probability $1/6^3$.

$$P(\text{player} = \text{yellow, private die} = \text{any, public die} = 2, \text{hit}) = 1/6^3 = \frac{1}{216}$$

and

$$P(\text{player} = \text{any color, private die} = \text{any, public die} = 2, \text{hit}) = \frac{6}{6^3} = \frac{1}{36}$$

Thus, we have the following result:

$$P(\text{player} = \text{yellow, private die} = \text{any} \mid \text{public die} = 2, \text{hit}) = \frac{\frac{1}{216}}{\frac{1}{36}} = \frac{1}{6}.$$

Why is the probability not higher, if we ruled out red and purple? The answer is that *P*(Blue = 1/6), *P*(Orange = 3/6), and *P*(Green = 1/6). So the probability of Orange increased dramatically, and *P*(Red = 0), and *P*(Purple = 0).

b) We use the "Miss" diagram in Fig. 4 to find

P(player = orange | public die = 2 & result = miss)

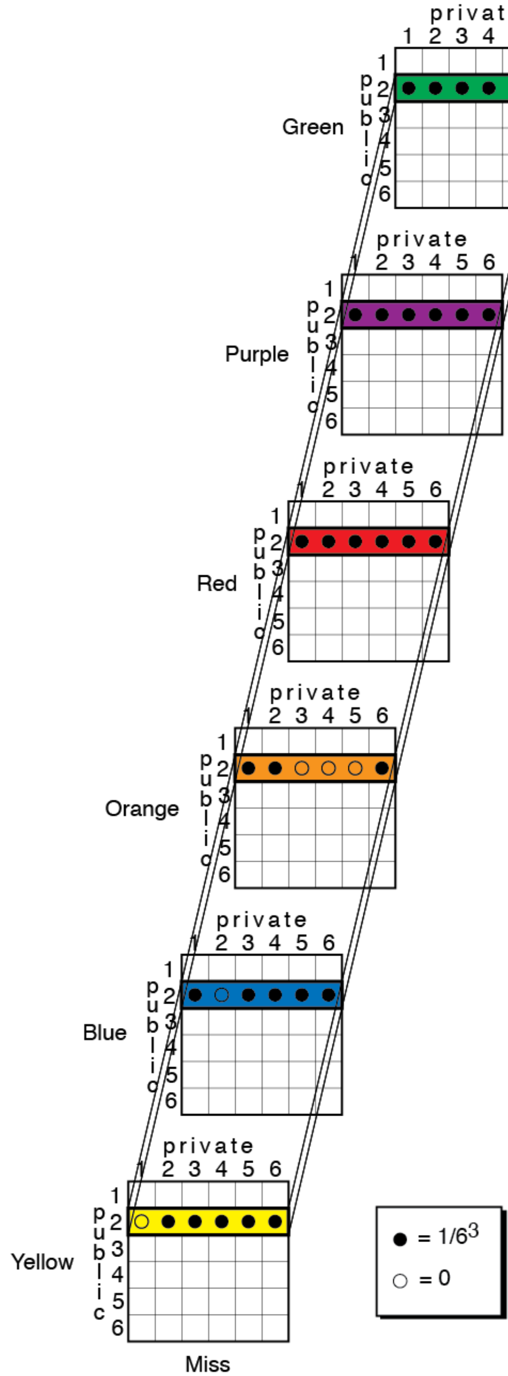

Fig. 4. Probability of public die = 2, private die, player color, and miss in spy game.

The calculations proceed as in (a).

$$P(\text{player} = \text{orange} \,|\, \text{public die} = 2, \text{miss})$$
$$= \frac{P(\text{player} = \text{orange}, \text{ private die} = \text{any}, \text{ public die} = 2, \text{miss})}{P(\text{player} = \text{any color}, \text{ private die} = \text{any}, \text{ public die} = 2, \text{hit})}$$

We count the dots that meet the given specifications, each of which has probability $1/6^3$.

$$P(\text{player} = \text{orange}, \text{ private die} = \text{any}, \text{ public die} = 2, \text{miss}) = 3/6^3 = \frac{1}{72}$$

and

$$P(\text{player} = \text{any color}, \text{ private die} = \text{any}, \text{ public die} = 2, \text{miss}) = \frac{30}{6^3} = \frac{5}{36}$$

Thus,

$$P(\text{player} = \text{orange, private die} = \text{any} \,|\, \text{public die} = 2, \text{miss}) = \frac{\frac{1}{72}}{\frac{5}{36}} = \frac{1}{10}.$$

The probability of Orange is significantly reduced but, as expected, is not zero.

c) For two turns, we have five independent variables:

- 1st public die number
- 1st private die number
- 2nd public die number
- 2nd private die number
- player color

These characteristics are independent. The value of each of them is not influenced by the values of the other two.

We have 6^5 equally likely outcomes, each with probability $1/6^5$. When we introduce the hit and miss results, we may reduce a 5-dimensional diagram to the outcomes shown in Fig. 5 below.

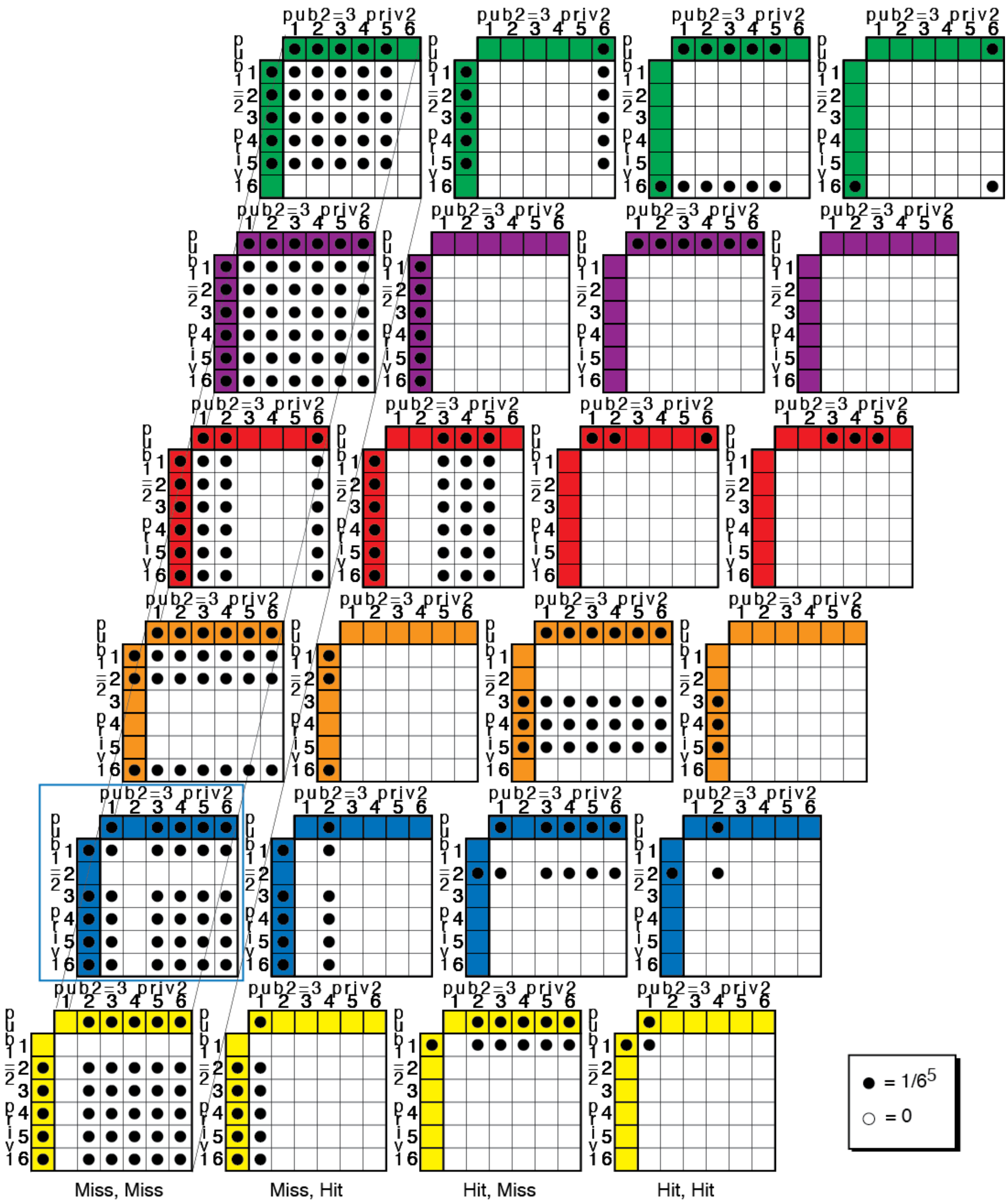

Fig. 5. Probability of 1st public die = 2, 1st private die, 2nd public die = 3, 2nd private die, player color, and hit or miss in spy game.

Our calculations:

$$P(\text{player blue} \,|\, \text{1st pub 2, 1st miss, 2nd pub 3, 2nd miss})$$
$$= \frac{P(\text{player blue, 1st pub 2, 1st priv any, 1st miss, 2nd pub 3, 2nd priv any, 2nd miss})}{P(\text{player any color, 1st pub 2, 1st priv any, 1st miss, 2nd pub 3, 2nd priv any, 2nd miss})}$$

For the numerator, we count the dots in the Miss, Miss column for color Blue.

$$P(\text{player blue, 1st pub 2, 1st priv any, 1st miss,}$$
$$\text{2nd pub 3, 2nd priv any, 2nd miss}) = \frac{25}{6^5}$$

For the denominator, we count the dots in the Miss, Miss column for all colors.

$$P(\text{player any color, 1st pub 2, 1st priv any, 1st miss,}$$
$$\text{2nd pub 3, 2nd priv any, 2nd miss}) = \frac{25+25+18+18+36+25}{6^5} = \frac{147}{6^5}$$

Taking the ratio, we have

$$P(\text{player blue} \,|\, \text{1st pub 2, 1st miss, 2nd pub 3, 2nd miss}) = \frac{25}{147}.$$

The probability that the player is Blue is slightly less than 1/6 after the first two turns, though only slightly.

If we repeated the calculation for Purple, we would find

$$P(\text{player purple} \,|\, \text{1st pub 2, 1st miss, 2nd pub 3, 2nd miss}) = \frac{36}{147}.$$

This probability is just slightly less than 1/4.