EX: Find the correlation, $\rho_{X\,Y}$, for the following joint probability density function:

$$f(x,y) = \begin{cases} \dfrac{x+y}{3} & 1 \le x \le 2 \text{ and } 1 \le y \le 2 \\ 0 & \text{otherwise} \end{cases}$$

The following information is given:

$$E(XY) = \frac{7}{10}$$

$$f_X(x) = \begin{cases} \dfrac{x}{3} + \dfrac{1}{2} & 1 \le x \le 2 \\ 0 & \text{otherwise} \end{cases}$$

$$f_Y(y) = \begin{cases} \dfrac{y}{3} + \dfrac{1}{2} & 1 \le y \le 2 \\ 0 & \text{otherwise} \end{cases}$$

$$\mu_X = \mu_Y = \frac{55}{36} \qquad \sigma_{XY} = \frac{-1}{1296}$$

SOL'N: We proceed with the calculation of variances.

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f_X(x)dx = \int_0^1 x^2\left(\frac{x}{3} + \frac{1}{2}\right)dx = \frac{29}{12}$$

Subtract the mean squared to find the variance:

$$\sigma_X^2 = E(X^2) - \mu_X^2 = \frac{29}{12} - \left(\frac{55}{36}\right)^2 = \frac{107}{1296}$$

By symmetry the variance is the same for Y.

Now we plug values into the formula for correlation:

$$\rho_{XY} \equiv \frac{\sigma_{XY}}{\sigma_X \sigma_Y} = \frac{-1/1296}{\sqrt{107/1296}\sqrt{107/1296}} = \frac{-1}{107}$$

The correlation is nonzero but very small.