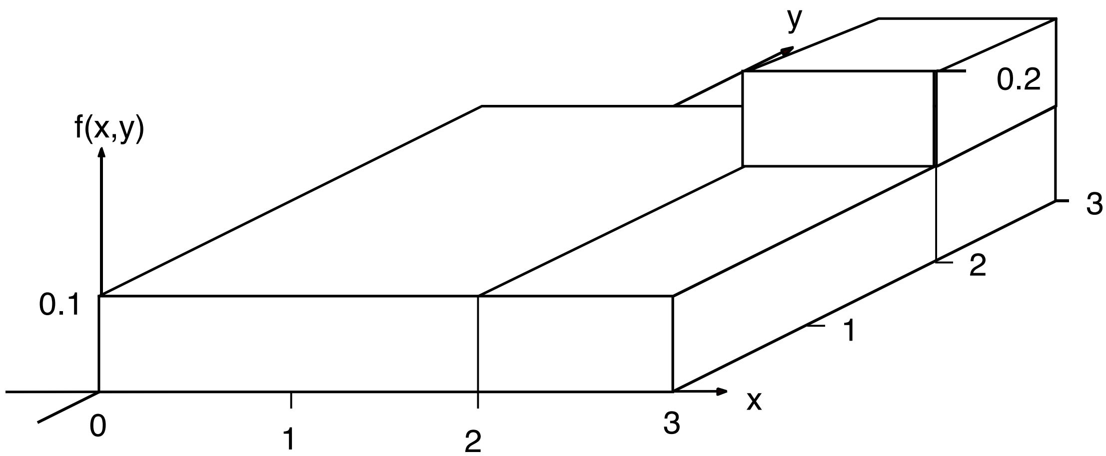
EX: Find the covariance, $\sigma_{X\,Y}$, for the joint probability density function shown below given $\mu_X = \mu_Y = 1.6$:



SOL'N: We may express $f(x, y)$ as a sum of two boxes:

$$f(x,y) = \begin{cases} 0.1 & 0 \le x \le 3 \text{ and } 0 \le y \le 3 \\ 0 & \text{otherwise} \end{cases}$$

$$+ \begin{cases} 0.1 & 2 \le x \le 3 \text{ and } 2 \le y \le 3 \\ 0 & \text{otherwise} \end{cases}$$

To find the covariance, we use the following formula:

$$\sigma_{XY} = E(XY) - \mu_X \mu_Y = E(XY) - 1.6^2 = E(XY) - 2.56$$

To find $E(XY)$, we compute a double integral:

$$E(XY) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xyf(x,y)dydx$$

or

$$E(XY) = \int_0^3 \int_0^3 xy \cdot 0.1dydx + \int_2^3 \int_2^3 xy \cdot 0.1dydx$$

or

$$E(XY) = 0.1\int_0^3 x \int_0^3 y \cdot dydx + 0.1\int_2^3 x \int_2^3 y \cdot dydx$$

or

$$E(XY) = 0.1\int_0^3 x \frac{y^2}{2}\Big|_{y=0}^{y=3} dx + 0.1\int_2^3 x \frac{y^2}{2}\Big|_{y=2}^{y=3} dx$$

or

$$E(XY) = 0.1\int_0^3 x \frac{9}{2} dx + 0.1\int_2^3 x\left(\frac{9}{2} - \frac{4}{2}\right) dx$$

or

$$E(XY) = 0.1 \frac{9}{2} \frac{x^2}{2}\Big|_{x=0}^{x=3} + 0.1 \frac{5}{2} \frac{x^2}{2}\Big|_{x=2}^{x=3}$$

or

$$E(XY) = 0.1 \cdot \frac{9}{2} \cdot \frac{9}{2} + 0.1 \cdot \frac{5}{2} \cdot \frac{5}{2} = 0.1 \cdot \frac{106}{4} = 2.65$$

Now subtract the product of the means.

$$\sigma_{XY} = E(XY) - \mu_X \mu_Y = 2.65 - 2.56 = 0.09$$