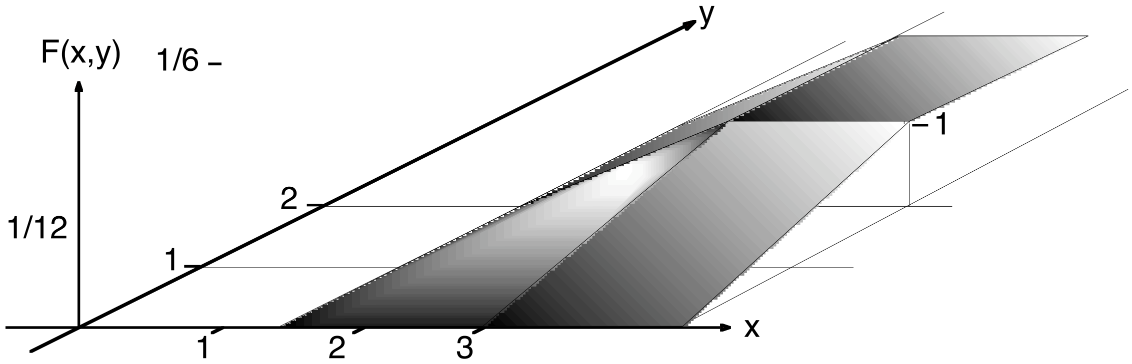
EX:



Plot the joint probability density function, $f(x, y)$, for the joint cumulative distribution function, $F(x, y)$, shown above in a cutaway view. Assume that X and Y are independent. The following information is also given:

$$F(x,y) = \begin{cases} 0 & x < 3/2 \text{ or } y < 0 \\ \dfrac{y}{3} & x = \dfrac{5}{2} \text{ and } 0 < y < 2 \\ \dfrac{x-3/2}{3} & \dfrac{3}{2} < x < 3 \text{ and } y = 1 \\ 1 & x > 3 \text{ and } y > 2 \end{cases}$$

SOL'N: $F(x, y)$ equals the volume of $f(x, y)$ to the left of x and in front of, (i.e., less than), y. Since $F(x, y) = 1$ for $x > 3$ and $y > 2$, all of the volume of $f(x, y)$ lies in the area where $x \le 3$ and $y \le 2$. Similarly, since $F(x, y) = 0$ for $x < 3/2$ or $y < 0$, all of the volume of $f(x, y)$ lies in the area where $x \ge 3/2$ and $y \ge 0$.

From the linear growth of $F(x, y)$ in the x direction in the area where $f(x, y)$ is nonzero and on the ramp for $3/2 \le x \le 3$ for $y > 2$, it follows that the area of the cross section of $F(x, y)$ in the y direction is constant as x changes. Similarly, from the linear growth of $F(x, y)$ in the y direction in the area where $f(x, y)$ is nonzero and on the ramp for $0 \le y \le 2$ for $x > 3$, it follows that the area of the cross section of $F(x, y)$ in the x direction is constant as y changes.

The simplest solution for $F(x, y)$ is a box of constant height over the region where $f(x, y)$ is nonzero. To achieve a volume of one, the height of the box should be 1/3:

$$f(x,y) = \begin{cases} \dfrac{1}{3} & \dfrac{3}{2} < x < 3 \text{ and } 0 < y < 2 \\ 0 & \text{otherwise} \end{cases}$$

If we integrate this $f(x, y)$, we get a complete description of $F(x, y)$:

$$F(x,y) = \begin{cases} \dfrac{x - 3/2}{3} \cdot y & \dfrac{3}{2} < x < 3 \text{ and } 0 < y < 2 \\ 0 & \text{otherwise} \end{cases}$$

Curiously, another surface also works. This surface has cross sections that have linear slopes on top in both the x and y directions.

$$f(x,y) = \begin{cases} \dfrac{2}{3}\left(\dfrac{x - \frac{3}{2}}{\frac{3}{2}}\right)\left(\dfrac{y}{2}\right) + \dfrac{2}{3}\left(\dfrac{3 - x}{\frac{3}{2}}\right)\left(\dfrac{2 - y}{2}\right) & \dfrac{3}{2} < x < 3 \text{ and } 0 < y < 2 \\ 0 & \text{otherwise} \end{cases}$$