EX: (This problem is motivated by problem of using the rand( ) function in Matlab® to create arbitrary probability density functions.) Given three independent random variables, V, W, and Z, that are uniformly distributed on [0, 1], describe a step-by-step calculation that yields random variables X and Y with the following joint density function (whose footprint is shaped like a diamond centered on the origin):

$$f(x,y) = \begin{cases} \frac{1}{2} & |X| + |Y| \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Hint: First generate X from the density function $f_X(x)$ using some simple algebra involving V and W. Then generate Y from the conditional probability density function $f(y \mid X)$. Use Z and some simple algebra to create Y.

SOL'N: The plot below shows the support (or footprint) of $f(x, y)$.

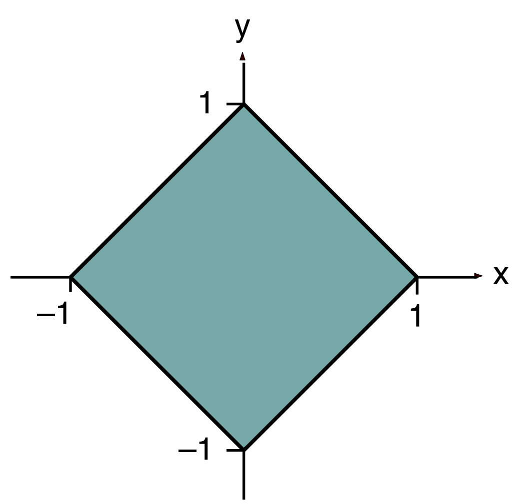

Fig. 1. Support (or footprint) of $f(x, y)$.

In a 3-dimensional view, the diamond shape of $f(x, y)$ has a constant height of 1/2.

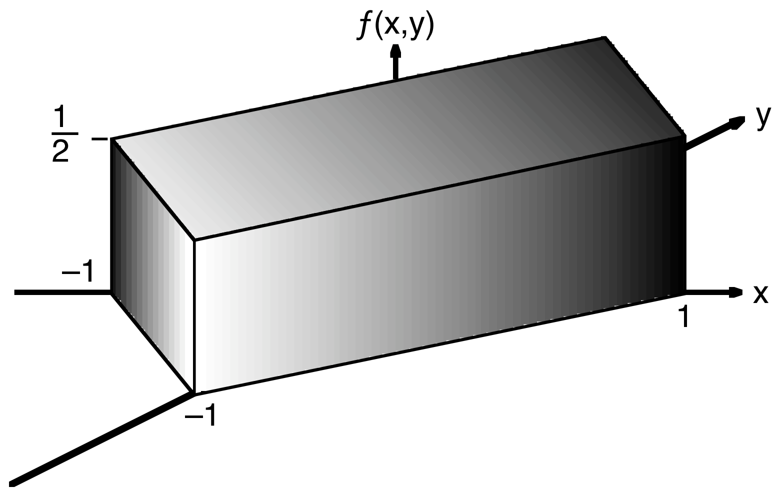

Fig. 2. 3-dimensional plot of $f(x, y)$.

The rationale for the hint is that we can write the joint probability, $f(x, y)$, as the product of a density function for x alone and a conditional probability for y given x:

$$f(x,y) = f(y \mid X = x) f_X(x)$$

This means that we can first pick X distributed as $f_X(x)$ and then pick Y distributed as $f(y \mid X)$.

To find $f_X(x)$, we use the standard formula for integration in the y direction:

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)dy$$

Fig. 3, below, shows the limits of the integral for a particular value of $x = x_o$ as the endpoints of a cross-section in the y direction. The value of $f(x, y)$ over this segment is one-half.

$$f_X(x_o) = \int_{-(1-|x_o|)}^{1-|x_o|} \frac{1}{2} dy = 1 - |x_o|$$

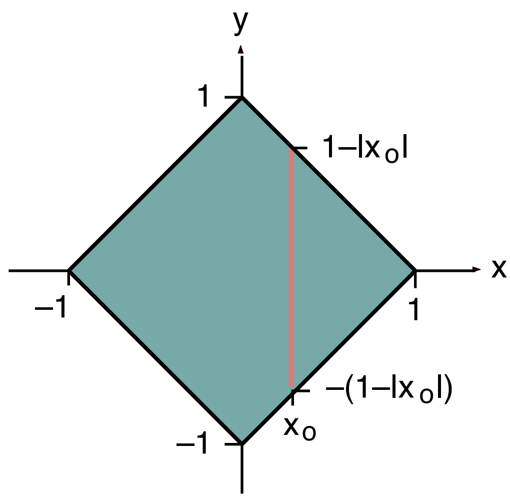

Fig. 3. Top view of cross-section used to calculate $f_X(x_o)$ and $f(y \mid X = x_o)$.

This above formula, written using absolute value, actually holds for any positive or negative value of x_o, and we have the following formula for probability density of X:

$$f_X(x) = 1 - |x|$$

Fig. 4 shows that $f_X(x)$ is triangular.

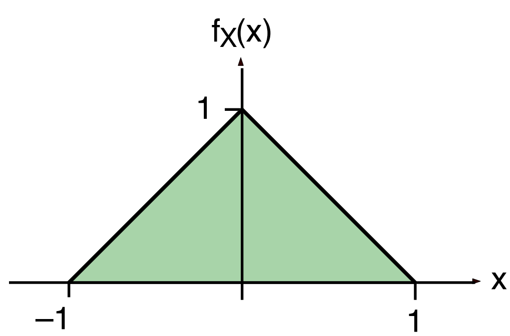

Fig. 4. Plot of $f_X(x)$.

There are two straightforward ways to generate a random variable, X, with this probability density function. The first is to add two uniformly distributed random variables together (and subtract one to give a mean of zero):

$$X = V + W - 1$$

The probability density function for X is computed as a convolution integral. We start with the probability density of V and find the probability density that $W = X - (V - 1)$. We integrate this product over possible values of V.

$$f_X(x) = \int_0^1 f_V(v) f_W(w = x - (v-1)) dv$$

We observe that $f_W(w) = 1$ when $0 < w < 1$.

$$f_X(x) = \int_0^1 f_V(v) \cdot \begin{cases} 1 & 0 < x - (v-1) < 1 \\ 0 & \text{otherwise} \end{cases} dv$$

Rearranging the inequality to express it in terms of v yields the following expression:

$$f_X(x) = \int_0^1 f_V(v) \cdot \begin{cases} 1 & x < v < x+1 \\ 0 & \text{otherwise} \end{cases} dv$$

Substituting $f_V(v) = 1$ and translating the expression for $f_W(w)$ into modifications of the limits of integration yields the following expression for the density function shown in Fig. 4:

$$f_X(x) = \int_{\max(0,x)}^{\min(1,x+1)} 1 \ dv = \begin{cases} \int_0^{x+1} 1 \ dv = x+1 & -1 < x < 0 \\ \int_x^1 1 \ dv = 1 - x & 0 < x < 1 \\ 0 & \textit{otherwise} \end{cases}$$

From the above discussion, the step-by-step procedure for calculating X is to use the following simple formula:

$$X = V + W - 1$$

Another way to obtain a random variable with the density function shown in Fig. 4 is to transform a single uniform random variable such as V by matching the cumulative distribution functions of X and V.

The cumulative distribution function for V is easily computed:

$$F_V(v) = \int_{-\infty}^{v} f_V(v)dv = \begin{cases} 0 & v < 0 \\ v & 0 < v < 1 \\ 1 & v > 1 \end{cases}$$

The cumulative distribution function for X is quadratic since $f_X(x)$ is linear.

$$F_X(x) = \int_{-\infty}^{\infty} f_X(x)dx = \begin{cases} 0 & x < -1 \\ \frac{1}{2}(x+1)^2 & -1 < x < 0 \\ 1 - \frac{1}{2}(x-1)^2 & 0 < x < 1 \\ 1 & x > 1 \end{cases}$$

Given a value for V, we find a value of X such that $F_X(x) = F_V(V)$. This translates in the following equation:

$$X \text{ satisfies } \begin{cases} \frac{1}{2}(X+1)^2 = V & 0 < V < \frac{1}{2} \\ 1 - \frac{1}{2}(X-1)^2 = V & \frac{1}{2} < V < 1 \end{cases}$$

or

$$X = \begin{cases} X = \sqrt{2V} - 1 & 0 < V < \frac{1}{2} \\ X = \sqrt{2(1-V)} + 1 & \frac{1}{2} < V < 1 \end{cases}$$

Now that we have X, we use the conditional probability density function, $f(y \mid X)$ for Y. We find $f(y \mid X)$ by first taking a cross section of $f(x, y)$ at $x = X$, as shown in Fig. 5.

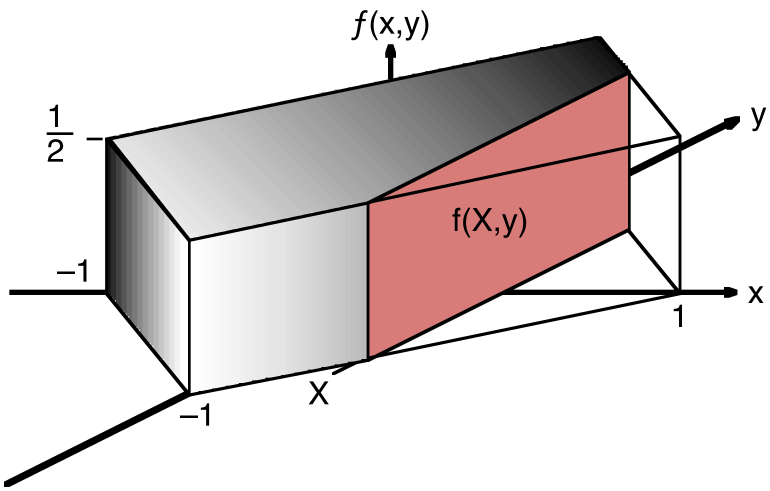

Fig. 5. Cross-section used to calculate $f_X(X)$ and $f(y \mid X)$.

We scale the cross section vertically so it will have a total area equal to one. Fig. 6 shows the result.

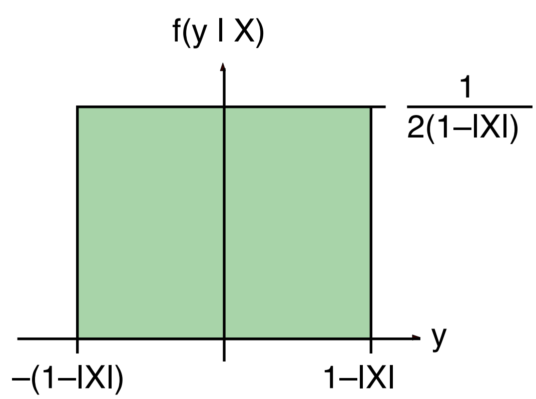

Fig. 6. Conditional probability $f(y \mid X)$.

We obtain this distribution by shifting and scaling a (0,1) uniform distribution such as Z.

$$Y = 2(1-|X|)\left(Z-\frac{1}{2}\right)$$