EX: The following formulas define the behavior of conditional probabilities:

$$P(A \mid B) = \frac{P(A,B)}{P(B)} \equiv \frac{P(A \text{ and } B)}{P(B)} \equiv \frac{P(A \cap B)}{P(B)} \qquad \text{(always true)}$$

$$P(A \mid B) = P(A) \qquad \text{(if } A \text{ and } B \text{ independent)}$$

$$P(A,B) = P(A)P(B) \qquad \text{(if } A \text{ and } B \text{ independent)}$$

For the following formulas, determine whether the formula is always true when A and B are independent.

a) $P(A \mid B)P(B \mid A) = P(A,B)$

b) $P(A' \mid B') = 1 - P(A \mid B)$

c) If $P(A) \neq 0$ and $P(B) \neq 0$, then $P(A,B) \neq 0$

d) For an arbitrary event, C, A is independent of $B \cap C$.

SOL'N: a) The first equation follows by direct application of the formulas for independent events:

$$P(A \mid B)P(B \mid A) = P(A)P(B) = P(A,B)$$

b) Because A and B are independent, we may immediately simplify the right-hand side of the equation:

$$1 - P(A \mid B) = 1 - P(A) = P(A')$$

Now we consider the left side of the equation:

$$P(A' \mid B') = \frac{P(A',B')}{P(B')} = \frac{P(A' \cap B')}{P(B')}$$

Using the Law to Total Probability, we may relate $P(A' \cap B')$ to $P(B')$:

$$P(A' \cap B') + P(A \cap B') = P(B')$$

or

$$P(A' \cap B') = P(B') - P(A \cap B')$$

Again using the Law of Total Probability, we may relate $P(A \cap B')$ to $P(A)$.

$$P(A \cap B') + P(A \cap B) = P(A)$$

or

$$P(A \cap B') = P(A) - P(A \cap B) = P(A) - P(A)P(B)$$

Substituting into our equation for $P(A' \cap B')$, we have the following result:

$$P(A' \cap B') = P(B') - (P(A) - P(A)P(B)) = (1 - P(B)) - (1 - P(B))P(A)$$

or

$$P(A' \cap B') = (1 - P(B))(1 - P(A)) = P(A')P(B')$$

The left side of the original equation now simplifies to $P(A')$:

$$P(A' | B') = \frac{P(A', B')}{P(B')} = \frac{P(A')P(B')}{P(B')} = P(A')$$

Thus, the left and right sides of the original equations are equal whenever A and B are independent:

$$P(A' | B') = 1 - P(A | B)$$

NOTE: Our derivation shows that, when A and B are independent events, we also have independence of A and B', A' and B, and A' and B'.

c) If $P(A) \neq 0$ and $P(B) \neq 0$, then $P(A,B) = P(A)P(B) \neq 0$ follows immediately. What is less immediately obvious is that this result implies that $P(A \cap B) \neq \varnothing$. In other words, the intersection of A and B is nonempty. Equivalently, A and B must overlap on a Venn diagram.

d) For an arbitrary event, C, we investigate the independence of A and $B \cap C$ by examining the conditional probability for A.

$$P(A | B \cap C) = \frac{P(A \cap B \cap C)}{P(B \cap C)}$$

Since $B \cap C$ may be any part of B, we may consider the case where $C = A \cap B$:

$$P(A \mid B \cap (A \cap B)) = \frac{P(A \cap B \cap (A \cap B))}{P(B \cap (A \cap B))} = \frac{P(A \cap B)}{P(A \cap B)} = 1$$

Since it is *not* always true that $P(A) = 1$, the equation in (d) is *not* always true.