EX: For the following joint probability density function, $f(x, y)$, are X and Y independent? (k is a scaling constant that makes the volume under $f(x, y)$ equal to one.) If X and Y are independent, find $f_X(x)$ and $f_Y(y)$.

$$f(x,y) = \begin{cases} k[\cos(\frac{\pi}{4}(x+y)) + \cos(\frac{\pi}{4}(x-y))] & 0 \le x \le 1 \text{ and } 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

SOL'N: In an attempt to write $f(x, y)$ as a product of $f_X(x)$ and $f_Y(y)$, we apply trigonometric identities to the formula for $f(x, y)$.

$$\cos(\tfrac{\pi}{4}(x+y)) = \cos(\tfrac{\pi}{4}x)\cos(\tfrac{\pi}{4}y) - \sin(\tfrac{\pi}{4}x)\sin(\tfrac{\pi}{4}y)$$

$$\cos(\tfrac{\pi}{4}(x-y)) = \cos(\tfrac{\pi}{4}x)\cos(\tfrac{\pi}{4}y) + \sin(\tfrac{\pi}{4}x)\sin(\tfrac{\pi}{4}y)$$

When we sum these identities, the sin( ) terms cancel out.

$$f(x,y) = \begin{cases} 2k[\cos(\frac{\pi}{4}x)\cos(\frac{\pi}{4}y)] & 0 \le x \le 1 \text{ and } 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

We now see that we can separate $f(x, y)$ into a product of functions of x and y.

Thus, X and Y are independent.

NOTE: In order to write $f(x, y)$ as a product of functions of x and y, we must also consider the support (or footprint) of $f(x, y)$ on the xy-plane. We must be able to separate a condition such as "$0 \le x \le 1$ and $0 \le y \le 1$" into a condition on x alone and a condition on y alone. The intersection of these conditions must yield the condition $0 \le x \le 1$ and $0 \le y \le 1$. Here, that is possible, since we can write $0 \le x \le 1$ for x and $0 \le y \le 1$ for y.

NOTE: When we define $f_X(x)$ and $f_Y(y)$, we must ensure that the total area under each is equal to one. Thus, we must include the appropriate amount of the scaling factor, $2k$, in $f_X(x)$ and $f_Y(y)$. By symmetry in the present problem, we use $\sqrt{2k}$ in each of $f_X(x)$ and $f_Y(y)$ so that the product of scaling factors is $2k$.

By symmetry, we have the following $f_X(x)$ and $f_Y(y)$ whose product is $f(x, y)$:

$$f_X(x) = \begin{cases} \sqrt{2k}\cos(\frac{\pi}{4}x) & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

$$f_Y(y) = \begin{cases} \sqrt{2k}\cos(\frac{\pi}{4}y) & 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

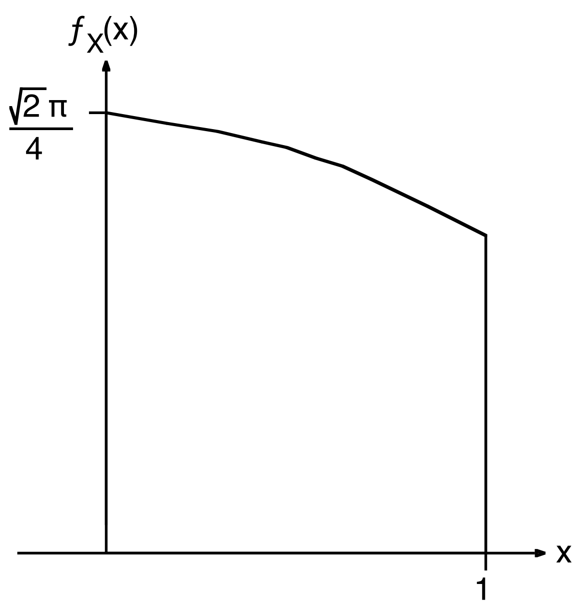
To find the value of k, we set the integral of $f_X(x)$ equal to one.

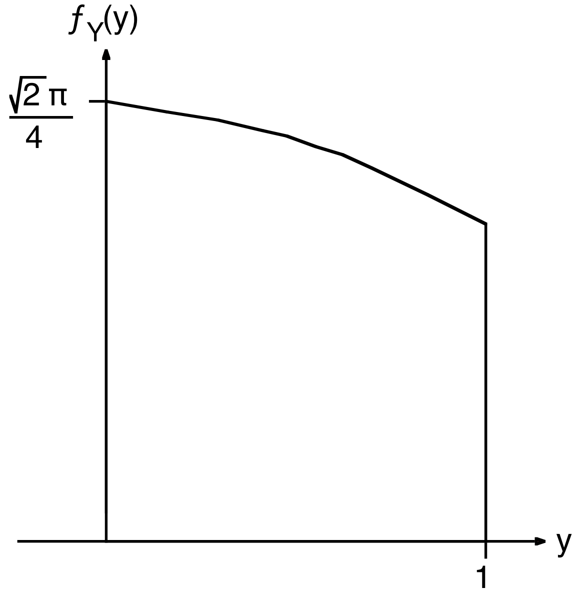
$$\int_0^1 \sqrt{2k}\cos(\tfrac{\pi}{4}x)dx = \sqrt{2k}\,\frac{4}{\pi}\sin(\tfrac{\pi}{4}x)\Big|_0^1 = \sqrt{2k}\,\frac{4}{\pi}\frac{1}{\sqrt{2}} = 1$$

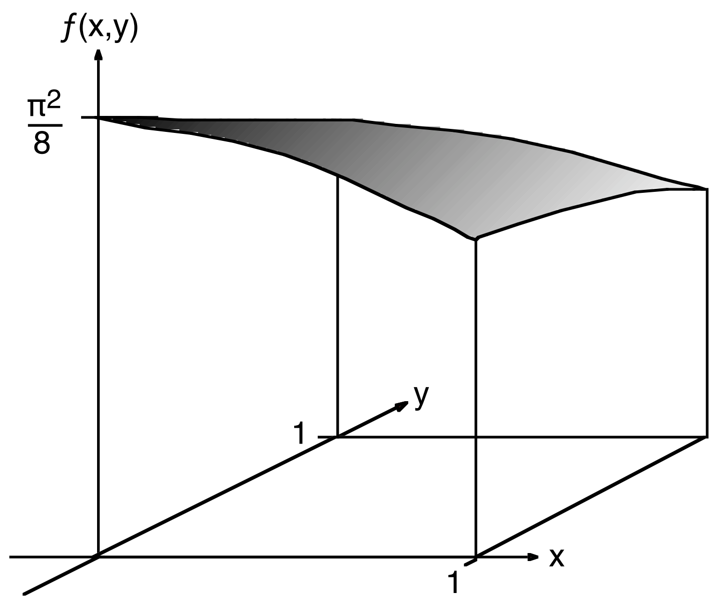
or

$$k = \frac{\pi^2}{16}$$

Plots of $f_X(x)$, $f_Y(y)$, and $f(x, y)$ are shown below.




$f_Y(y)$
$\frac{\sqrt{2}\,\pi}{4}$
y
1


$f(x,y)$
$\frac{\pi^2}{8}$
y
1
x
1