EX: An engineer is measuring the illumination, Z, at one point in an optical circuit. The engineer knows that Z is a linear combination of illumination values, X and Y, at two other points in the circuit. In other words, $Z = aX + bY$. The engineer has gathered the following information about the illumination values:

$$\mu_X = 6 \qquad \mu_Y = 15 \qquad \mu_Z = 6.5$$

$$\sigma_X^2 = 4 \qquad \sigma_Y^2 = 9 \qquad \sigma_{XY} = 3 \qquad \sigma_Z^2 = \frac{7}{4}$$

Find the values of a and b, (assuming they are positive).

SOL'N: We use the following tools for linear combinations of random variables:

$$\mu_Z \equiv \mu_{aX+bY} = a\mu_X + b\mu_Y$$
$$\sigma_Z^2 \equiv \sigma_{aX+bY}^2 = a^2\sigma_X^2 + b^2\sigma_Y^2 + 2ab\sigma_{XY}$$

Substituting values given in the problem, we have the following two equations with unknowns a and b:

$$6.5 = a \cdot 6 + b \cdot 15$$
$$\frac{7}{4} = a^2 \cdot 4 + b^2 \cdot 9 + 2ab \cdot 3$$

We can solve the first equation for b and substitute the result into the second equation:

$$b = \frac{6.5 - 6a}{15}$$

When we substitute for b, the second equation becomes a quadratic equation in a:

$$\frac{7}{4} = a^2 \cdot 4 + \left(\frac{6.5 - 6a}{15}\right)^2 \cdot 9 + 2a\left(\frac{6.5 - 6a}{15}\right) \cdot 3$$

Multiplying both sides by 30^2 and combining terms gives the following quadratic equation:

$$2736a^2 - 468a - 54 = 0$$

The solutions are $a = 1/4$ or $a = -3/38$. We use the positive root, $a = 1/4$, as instructed in the problem.

Returning to the equation relating b to a, we get the following value:

$$b = \frac{6.5 - 6a}{15} = \frac{6.5 - 6(1/4)}{15} = \frac{1}{3}$$