EX: Three random variables, X_1, X_2, and X_3, are independent and uniformly distributed on [–1/512, 1/512]. (They represent the distributions of errors for an 8-bit analog-to-digital converter that quantizes voltages by rounding off to the nearest $1/2^8$ V.) Find the mean and variance of $Z = (X_1 + X_2 + X_3)/3$. (This represents the result of oversampling by a factor of three and averaging, as compared to using a single sample.)

SOL'N: We use the following tools for linear combinations of random variables (extended from formulas for the sum of two random variables):

$$\mu_Z \equiv \mu_{aX_1+bX_2+cX_3} = a\mu_{X_1} + b\mu_{X_2} + c\mu_{X_3}$$

$$\sigma^2_{aX+b} = a^2\sigma^2_X$$

and, (for independent random variables)

$$\sigma^2_Z \equiv \sigma^2_{aX_1+bX_2+cX_3} = a^2\sigma^2_{X_1} + b^2\sigma^2_{X_2} + c^2\sigma^2_{X_3}$$

First, we observe that the means of X_1, X_2, and X_3 are zero since they are uniform distributions centered at $x = 0$. Thus, we can calculate the mean of Z as follows:

$$\mu_Z \equiv \frac{1}{3}\mu_{X_1} + \frac{1}{3}\mu_{X_2} + \frac{1}{3}\mu_{X_3} = 0$$

Second, we observe that one may obtain each of the X_i from a uniform distribution on (0,1) that is shifted by –1/2, (which doesn't change the variance), and then scaling by $a = 1/256$. Since the variance for a uniform distribution on (0,1) is 1/12, we have the following variance for each of the X_i:

$$\sigma^2_{X_i} = a^2\sigma^2_{\mathrm{u}(0,1)} = \left(\frac{1}{256}\right)^2 \frac{1}{12} = \frac{1}{786{,}432}$$

Now we apply the formula for the variance of a sum of independent random variables to Z where $a = b = c = 1/3$:

$$\sigma^2_Z \equiv \sigma^2_{aX_1+bX_2+cX_3} = a^2\sigma^2_{X_1} + b^2\sigma^2_{X_2} + c^2\sigma^2_{X_3}$$

or

$$\sigma_Z^2 \equiv \sigma_{X_1/3+bX_2/3+cX_3/3}^2 = \left(\frac{1}{3}\right)^2 \frac{1}{256^2 12} + \left(\frac{1}{3}\right)^2 \frac{1}{256^2 12} + \left(\frac{1}{3}\right)^2 \frac{1}{256^2 12}$$

or

$$\sigma_Z^2 \equiv \sigma_{X_1/3+bX_2/3+cX_3/3}^2 = \frac{3}{9 \cdot 256^2 \cdot 12} = \frac{1}{2{,}359{,}296}$$

Note that we have a factor of 1/9 for each term but we have 3 terms. Thus, our final answer contains only a factor of 3 in the denominator.