TOOL: Given probability density function, $f_X(x)$, for X, the probability density function (pdf), $f_Y(y)$ for $Y = aX + b$, $(a \neq 0)$,

is

$$f_Y(y) = \frac{1}{|a|} f_X\left(x = \frac{y-b}{a}\right).$$

Also, the mean and variance transform as follows:

$$\mu_Y = a\mu_X + b \qquad \sigma_Y^2 = a^2\sigma_X^2.$$

PROOF: By definition, $f_Y(y)$ is the derivative of the cumulative probability distribution function.

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy} P(Y \leq y)$$

Making a direct substitution for Y, we have an expression that we can transform into a statement about the probability of X:

$$\frac{d}{dy} P(Y \leq y) = \frac{d}{dy} P(aX + b \leq y) = \begin{cases} \frac{d}{dy} P\left(X \leq \frac{y-b}{a}\right) & a > 0 \\ \frac{d}{dy} P\left(X \geq \frac{y-b}{a}\right) & a < 0 \end{cases}$$

The last expressions are statements about the cumulative distribution function of X.

$$f_Y(y) = \begin{cases} \frac{d}{dy} F_X\left(x = \frac{y-b}{a}\right) & a > 0 \\ \frac{d}{dy}\left[1 - F_X\left(x = \frac{y-b}{a}\right)\right] & a < 0 \end{cases}$$

Using the chain rule from calculus, it is possible to write the above derivatives in terms of x:

$$f_Y(y) = \begin{cases} \frac{d}{dx} F_X\left(x = \frac{y-b}{a}\right)\frac{dy}{dx} & a > 0 \\ \frac{d}{dx}\left[1 - F_X\left(x = \frac{y-b}{a}\right)\right]\frac{dy}{dx} & a < 0 \end{cases}$$

The derivatives in terms of x are probability density functions for X, and the derivatives of $y = ax + b$ are equal to a:

$$f_Y(y) = \begin{cases} f_X\left(x = \dfrac{y-b}{a}\right)a & a > 0 \\ -f_X\left(x = \dfrac{y-b}{a}\right)a & a < 0 \end{cases}$$

This may be written more compactly as follows:

$$f_Y(y) = \frac{1}{|a|} f_X\left(x = \frac{y-b}{a}\right), \qquad a \neq 0$$

For the mean of Y, we write the integral formula:

$$\mu_Y = E(aX + b) = \int_{-\infty}^{\infty} (ax + b) f_X(x)dx$$

We rewrite the integral in two parts and exploit the property that the area under the pdf is equal to one:

$$\mu_Y = a\int_{-\infty}^{\infty} x f_X(x)dx + b\int_{-\infty}^{\infty} f_X(x)dx = a\mu_X + b$$

For the variance, we substitute for Y in the variance formula:

$$\sigma_Y^2 = E([Y - \mu_Y]^2) = E([aX + b - (a\mu_X + b)]^2)$$

or

$$\sigma_Y^2 = E(a^2[X - \mu_X]^2) = a^2 E([X - \mu_X]^2) = a^2\sigma_X^2$$