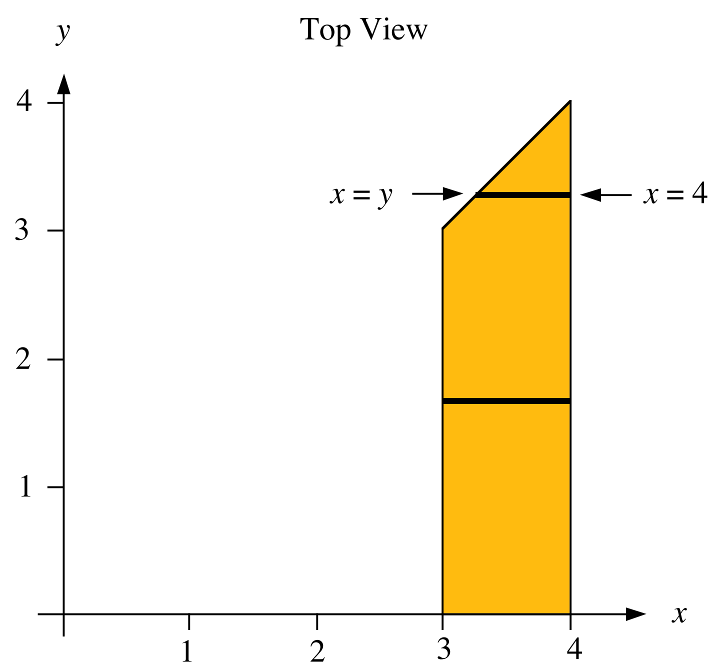
EX: A joint probability density function is defined as follows:

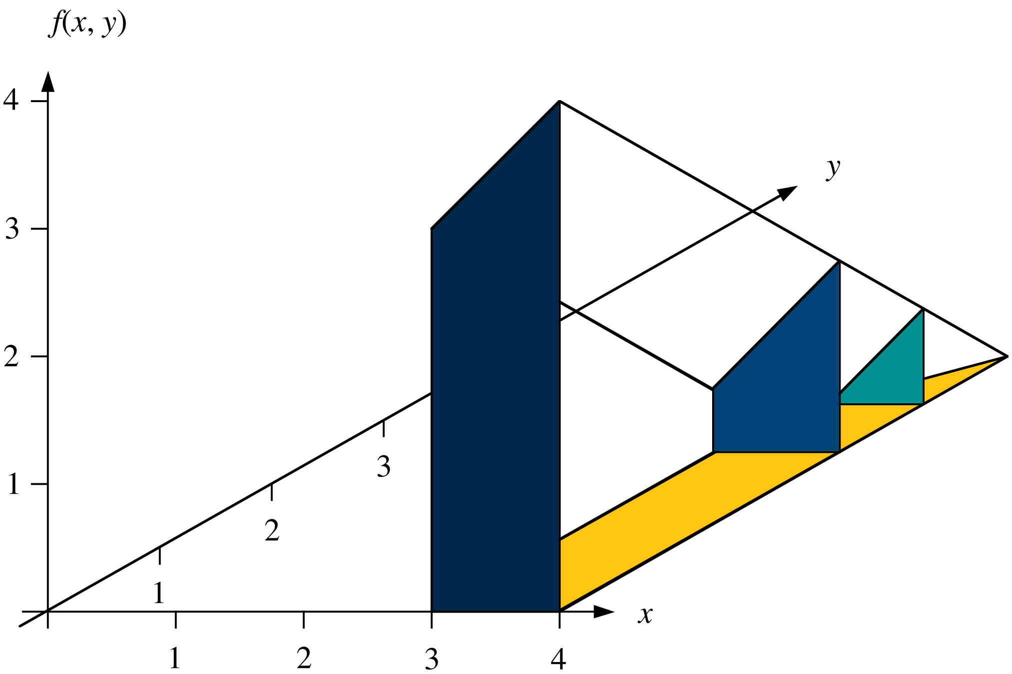
$$f(x,y) = \begin{cases} x - y & 3 \le x \le 4 \text{ and } 0 \le y \le x \\ 0 & \text{otherwise} \end{cases}$$

Find the marginal probability density function, $f_Y(y)$. Note that the condition $0 \le y \le x$ depends on x.

SOL'N: The region, $3 \le y \le 4$ and $0 \le y \le x$, on which $f(x, y) \ne 0$ is the support of $f(x, y)$. It is a trapezoid, as shown below. The diagram also shows several horizontal segments over which $f(x, y) \ne 0$ as a function of position y.



The illustration, below, shows the 3-dimensional shape of $f(x, y)$. The figure also shows cross-sections in the x direction. The value of $f_Y(y)$ at any value of y is equal to the area of the cross-section of $f(x, y)$ in the x direction at that value of y.



We use the diagram of the support (or footprint) of $f(x, y)$ to determine the limits of integration in the following calculation that determines the area of the cross-section.

$$f_Y(y) = \begin{cases} \int_{x=y}^{x=4} f(x,y)\,dx & 3 \le y \le 4 \\ \int_{x=3}^{x=4} f(x,y)\,dx & 0 \le y \le 3 \\ 0 & \text{otherwise} \end{cases}$$

Substituting $f(x, y) = x - y$ into the integral, we evaluate $f_Y(y)$.

$$f_Y(y) = \begin{cases} \int_{x=y}^{x=4} (x-y)\,dx & 3 \le y \le 4 \\ \int_{x=3}^{x=4} (x-y)\,dx & 0 \le y \le 3 \\ 0 & \text{otherwise} \end{cases}$$

or

$$f_Y(y) = \begin{cases} \left(\frac{x^2}{2} - xy\right)\Big|_{x=y}^{x=4} & 3 \le y \le 4 \\ \left(\frac{x^2}{2} - xy\right)\Big|_{x=3}^{x=4} & 0 \le y \le 3 \\ 0 & \text{otherwise} \end{cases}$$

or

$$f_Y(y) = \begin{cases} \frac{16 - y^2}{2} - (4 - y)y & 3 \le y \le 4 \\ \frac{16 - 9}{2} - (4 - 3)y & 0 \le y \le 3 \\ 0 & \text{otherwise} \end{cases}$$

or

$$f_Y(y) = \begin{cases} 8 - 4y + \frac{y^2}{2} & 3 \le y \le 4 \\ \frac{7}{2} - y & 0 \le y \le 3 \\ 0 & \text{otherwise} \end{cases}$$