EX: Prove the following:

a) $E(X)$ = center of mass of the density function, $f(x)$. That is, show

$$\int_{-\infty}^{\infty} [x - E(X)] f(x) dx = 0$$

b) $E(X)$ may be computed from centers of mass of pieces concentrated at single points. In other words, show that, for segments that cover $x = (-\infty, \infty)$ without overlapping, we may reduce the "mass" of $f(x)$ to single points:

$m_i = \int_{x_i}^{x_{i+1}} f(x) dx$ defines mass for ith segment of $f(x)$

$\int_{x_i}^{x_{i+1}} [x - c_i] f(x) dx = 0$ defines center of mass for ith segment of $f(x)$

$E(X) = \dfrac{\sum_{i=0}^{N} c_i m_i}{\sum_{i=0}^{N} m_i}$, or $E(X) = \sum_{i=0}^{N} c_i m_i$ since total probability (mass) = 1.

PF: a) We break the integral into pieces and observe that E(X) is a constant that we may take outside the integral:

$$\int_{-\infty}^{\infty} [x - E(X)] f(x) dx = \int_{-\infty}^{\infty} x f(x) dx - \int_{-\infty}^{\infty} E(X) f(x) dx$$

$$= E(X) - \int_{-\infty}^{\infty} E(X) f(x) dx = E(X) - E(X) \int_{-\infty}^{\infty} f(x) dx$$

Since the total probability is equal to unity, the last integral has a value of unity and we obtain a value of zero, as desired:

$$\int_{-\infty}^{\infty} [x - E(X)] f(x) dx = E(X) - E(X) = 0$$

b) Given $\int_{x_i}^{x_{i+1}} [x - c_i] f(x) dx = 0$, we have

$$\int_{x_i}^{x_{i+1}} x f(x) dx = \int_{x_i}^{x_{i+1}} c_i f(x) dx = c_i \int_{x_i}^{x_{i+1}} f(x) dx.$$

Thus, $c_i = \dfrac{\int_{x_i}^{x_{i+1}} x f(x) dx}{\int_{x_i}^{x_{i+1}} f(x) dx}$. We substitute this and the definition for m_i into the center of mass formula:

$$\sum_{i=0}^{N} c_i m_i = \sum_{i=0}^{N} \frac{\int_{x_i}^{x_{i+1}} xf(x)dx}{\int_{x_i}^{x_{i+1}} f(x)dx} \cdot \int_{x_i}^{x_{i+1}} f(x)dx = \sum_{i=0}^{N} \int_{x_i}^{x_{i+1}} xf(x)dx$$

The expression on the right is just the integral from $-\infty$ to ∞ broken into N pieces that are then put back together by the summation. Thus, we have

$\sum_{i=0}^{N} c_i m_i = \int_{-\infty}^{\infty} xf(x)dx \equiv E(X)$, and the proof is complete.