EX: In communications theory, a transmitted bit value with additive noise is often modeled as a one or zero plus a gaussian distributed random variable. In a certain quadrature modulation scheme, transmitted pairs of bits may be treated as points at (–1, –1), (–1, 1), (1, –1), and (1, 1) plus an ordered pair of random variables, X and Y, drawn from a two-dimensional gaussian distribution with correlation ρ_{XY} and zero means:

$$f(x,y)=\frac{1}{2\pi\sqrt{1-\rho_{XY}^2}}e^{-(x^2-2\rho_{XY}\cdot xy+y^2)/2(1-\rho_{XY}^2)}$$

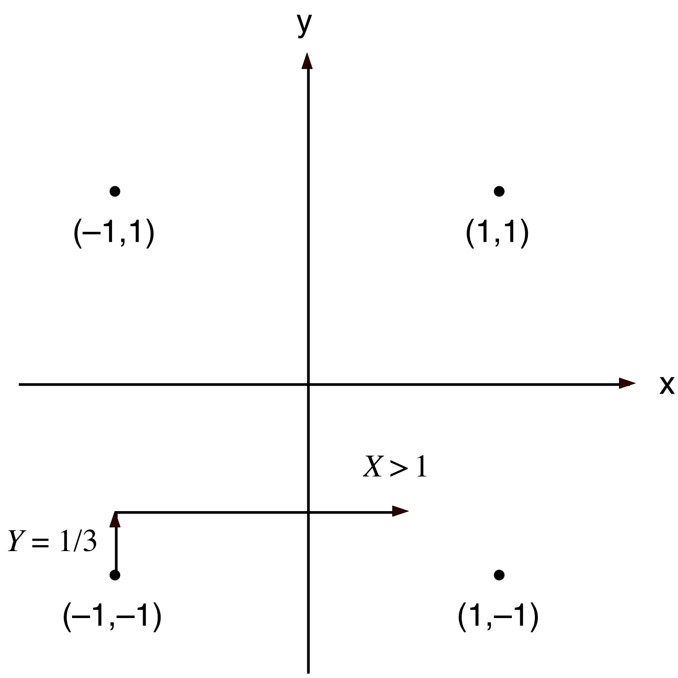
where

$$\rho_{XY}=\frac{\sigma_{XY}}{\sigma_X\sigma_Y}=\frac{E\{XY\}-\mu_X\mu_Y}{\sqrt{E\{X^2\}-\mu_X^2}\cdot\sqrt{E\{Y^2\}-\mu_Y^2}}=\frac{1}{2}$$

To determine which pair of bits is sent, one computes $f(x, y)$ for each of the points at (–1, –1), (–1, 1), (1, –1), and (1, 1). The point with the largest value of $f(x, y)$ is taken as the bit pair that was originally sent. It turns out that the point with the largest value of $f(x, y)$ is the point closest to (X, Y).

Normally, X and Y are unknown, but suppose that a measurement of Y is available, and suppose that $Y = 1/3$. If a (–1, –1) is transmitted, find the probability that $(-1 + X, -1 + Y)$ is closer to (1, –1) than to (–1, –1).

SOL'N: We consider a diagram showing the signals and noise.



For the received signal to be closer to (1, –1) than to (–1, –1), the value of X must be greater than 1, as the diagram shows. Thus, we find the conditional probability $f(x \mid y = 1/3)$ and determine the conditional probability that $X > 1$.

$$f(x \mid y = 1/3) = \frac{f(x, y = 1/3)}{\int_{-\infty}^{\infty} f(x, y = 1/3)dx}$$

We compute the integral in the denominator by factoring out the integral of a gaussian distribution whose value must be equal to one. Our first step in this process is to substitute numerical values for ρ and y.

$$\int_{-\infty}^{\infty} f(x, y = 1/3)dx = \int_{-\infty}^{\infty} \frac{1}{2\pi\sqrt{1 - \frac{1}{4}}} e^{-\left(x^2 - 2\frac{1}{2}\cdot x\left(\frac{1}{3}\right) + \left(\frac{1}{3}\right)^2\right)/2\left(1 - \frac{1}{4}\right)} dx$$

or

$$\int_{-\infty}^{\infty} f(x,y=1/3)dx = \int_{-\infty}^{\infty} \frac{1}{2\pi\sqrt{\frac{3}{4}}} e^{-\left(x^2 - 2\frac{1}{6}\cdot x + \frac{1}{9}\right)/2\left(\frac{3}{4}\right)} dx$$

Now we complete the square in the exponent so we will have something that is of the form $(x - \mu_X)^2$. We add and subtract terms of the same value. The extra constants that are not part of the square in the exponent may be extracted as multiplicative factors, since exponents add.

$$\int_{-\infty}^{\infty} f(x,y=1/3)dx = \int_{-\infty}^{\infty} \frac{1}{2\pi\sqrt{\frac{3}{4}}} e^{-\left(x^2 - 2\frac{1}{6}\cdot x + \left(\frac{1}{6}\right)^2 - \left(\frac{1}{6}\right)^2 + \frac{1}{9}\right)/2\left(\frac{3}{4}\right)} dx$$

or

$$\int_{-\infty}^{\infty} f(x,y=1/3)dx = \int_{-\infty}^{\infty} \frac{1}{2\pi\sqrt{\frac{3}{4}}} e^{-\left(x - \frac{1}{6}\right)^2/2\left(\frac{3}{4}\right)} \cdot e^{\left(\left(\frac{1}{6}\right)^2 - \frac{1}{9}\right)/2\left(\frac{3}{4}\right)} dx$$

We now take the constant exponential terms outside of the integral, and we also extract a factor of the square root of 2π, leaving the integral of a gaussian distribution whose mean is 1/6 and whose variance is 3/4:

$$\int_{-\infty}^{\infty} f(x,y=1/3)dx = \frac{e^{\left(\left(\frac{1}{6}\right)^2 - \frac{1}{9}\right)/2\left(\frac{3}{4}\right)}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\frac{3}{4}}} e^{-\left(x - \frac{1}{6}\right)^2/2\left(\frac{3}{4}\right)} \cdot dx$$

or

$$\int_{-\infty}^{\infty} f(x,y=1/3)dx = \frac{e^{\frac{-1}{18}}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma_X^2}} e^{-(x-\mu_X)^2/2\sigma_X^2} \cdot dx$$

where

$\mu_X = 1/6$ and $\sigma_X^2 = 3/4$

The value of the integral of the gaussian is one, and we have the final value we are seeking:

$$\int_{-\infty}^{\infty} f(x, y = 1/3)dx = \frac{e^{\frac{-1}{18}}}{\sqrt{2\pi}}$$

Now we evaluate the conditional probability we originally set out to find:

$$f(x \mid y = 1/3) = \frac{f(x, y = 1/3)}{\int_{-\infty}^{\infty} f(x, y = 1/3)dx} = \frac{\frac{1}{2\pi\sqrt{\frac{3}{4}}} e^{-\left(x^2 - 2\frac{1}{6}\cdot x + \frac{1}{9}\right)/2\left(\frac{3}{4}\right)}}{\frac{e^{\frac{-1}{18}}}{\sqrt{2\pi}}}$$

If we complete the square in the exponent, as before, we get

$$f(x \mid y = 1/3) = \frac{\frac{e^{\frac{-1}{18}}}{\sqrt{2\pi}} \cdot \frac{1}{\sqrt{2\pi\frac{3}{4}}} e^{-\left(x - \frac{1}{6}\right)^2/2\left(\frac{3}{4}\right)}}{\frac{e^{\frac{-1}{18}}}{\sqrt{2\pi}}}$$

or

$$f(x \mid y = 1/3) = \frac{1}{\sqrt{2\pi\frac{3}{4}}} e^{-\left(x - \frac{1}{6}\right)^2/2\left(\frac{3}{4}\right)}$$

NOTE: The mean is $\mu = \rho y$ and the variance is $\sigma^2 = 1 - \rho^2$. This will always be the case, and we may dispense with the lengthy calculation.

Now we find P(X > 1). This is equal to $1 - F(x = 1)$ where $F(x)$ is the cumulative distribution for $f(x \mid y = 1/3)$. We use the formula that converts the value of x to the value of z for a standard gaussian random variable:

$$z = \frac{x - \mu}{\sigma}$$

Using the mean and variance of our conditional probability and $X = 1$, we have the following:

$$z = \frac{1 - \frac{1}{6}}{\sqrt{\frac{3}{4}}} = \frac{5}{3\sqrt{3}} \approx 0.962$$

From a table of values for the cumulative distribution, $F(z)$, of a standard gaussian, we obtain the value $F(0.962) = 0.832$. Thus, our final answer is as follows:

$$P(X > 1) = 1 - 0.832 = 0.168$$