EX: The probability density function for a 2-dimensional gaussian (or normal) distribution is described by the following formula:

$$f(x,y) = \frac{1}{2\pi\sqrt{1-\rho_{XY}^2}} e^{-(x^2 - 2\rho_{XY}\cdot xy + y^2)/2(1-\rho_{XY}^2)}$$

where $\rho_{XY} = 1/2$

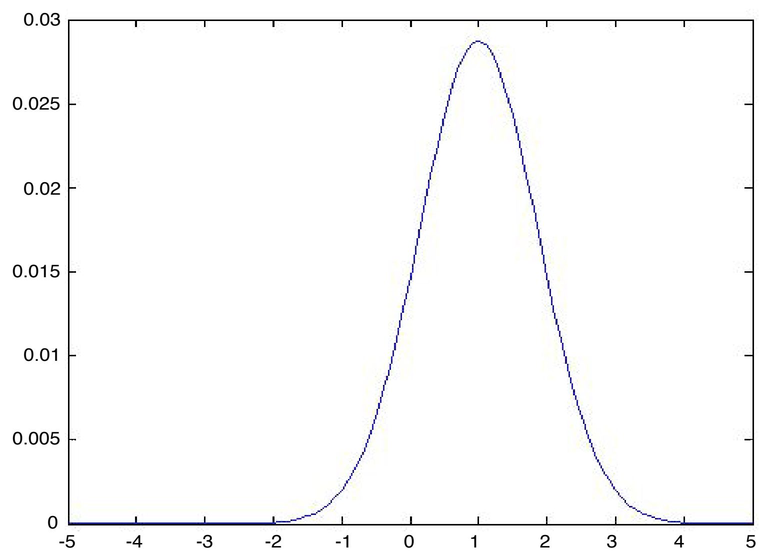
a) Make a (1-dimensional) plot of a cross-section of $f(x, y)$ on the line $x = 2$ (while y varies from $-\infty$ to ∞). Describe the shape of this curve. Determine whether this curve is a valid probability density function.

b) Make a (1-dimensional) plot of a cross-section of $f(x, y)$ on the line $y = 2x$. Describe the shape of this curve. Determine whether this curve is a valid probability density function.

SOL'N: a) The function we are plotting is $f(2, y)$:

$$f(2,y) = \frac{1}{2\pi\sqrt{1-\frac{1}{4}}} e^{-(2^2 - 2\frac{1}{2}\cdot 2y + y^2)/2(1-\frac{1}{4})} = \frac{1}{\pi\sqrt{3}} e^{-(4-2y+y^2)/\frac{3}{2}}$$

The plot of $f(2, y)$ shown below was generated with Matlab® code:



The shape of $f(2,y)$ is similar to a gaussian distribution. For it to be a valid pdf, it must have a total area equal to one. We cannot integrate $f(2,y)$

directly, but we can use the method of completing the square to write the exponent as y minus a constant—corresponding to the mean value of y—squared over a constant—corresponding to the variance.

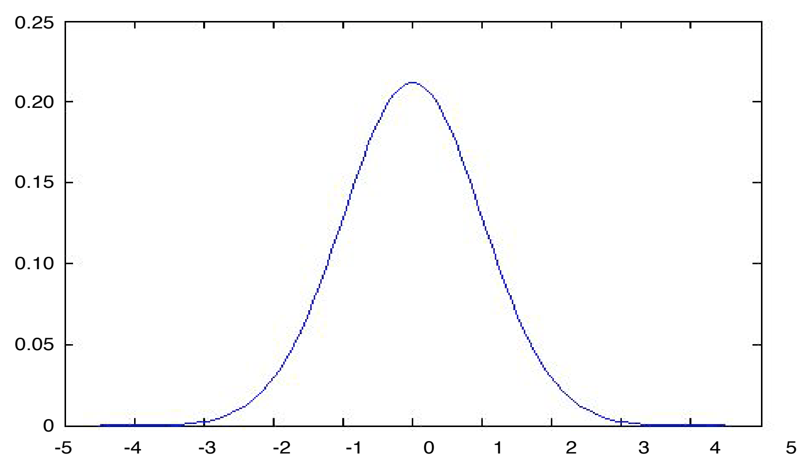
$$f(2,y) = \frac{1}{\pi\sqrt{3}} e^{-\left[(y-1)^2+3\right]/\frac{3}{2}} = \frac{1}{\pi\sqrt{3}} e^{-(y-1)^2/\frac{3}{2}} e^{-2}$$

To achieve the desired form, we extract a factor of e^{-2} from the exponential. Now we can write $f(2,y)$ in the form of a gaussian multiplied by a constant.

$$f(2,y) = \frac{e^{-2}}{\sqrt{2\pi}} \frac{1}{\sqrt{2\pi\frac{3}{4}}} e^{-(y-1)^2/2\cdot\frac{3}{4}}$$

The constant multiplying the gaussian in this case is $e^{-2}/\sqrt{2\pi}$. Thus, the area under $f(2,y)$ is $e^{-2}/\sqrt{2\pi}$ rather than 1. Thus, this slice of the 2-dimensional gaussian is not a gaussian.

b) The plot in this second case requires a bit more work. To achieve the correct scale, the distance from the original must be faithfully preserved. Using a parameterized curve, we achieve the desired result. Let t be the distance from the origin along the line $y = 2x$. If $t = 1$, then the Pythagorean theorem dictates that $x = 1/\sqrt{5}$ and $y = 2/\sqrt{5}$. In general, we have $x = t/\sqrt{5}$ and $y = 2t/\sqrt{5}$. Using these values, we obtain the plot shown below, generated in Matlab®.



As before, we can write $f(x = t/\sqrt{5}, y = 2t/\sqrt{5})$ in terms of a gaussian distribution.

$$f\left(\frac{t}{\sqrt{5}}, \frac{2t}{\sqrt{5}}\right) = \frac{1}{2\pi\sqrt{1-\frac{1}{4}}} e^{-(t^2 - 2\frac{1}{2}\cdot 4t^2 + (2t)^2)/5\cdot 2(1-\frac{1}{4})}$$

or

$$f\left(\frac{t}{\sqrt{5}}, \frac{2t}{\sqrt{5}}\right) = \frac{1}{2\pi\sqrt{\frac{3}{4}}} e^{-t^2/2\cdot\frac{15}{4}} = \sqrt{\frac{5}{2\pi}} \frac{1}{\sqrt{2\pi\frac{15}{4}}} e^{-t^2/2\cdot\frac{15}{4}}$$

We have a gaussian distribution multiplied by $\sqrt{5/2\pi}$. Thus, the area under the curve is $\sqrt{5/2\pi}$ rather than 1, and the curve is not a valid pdf.