DERIV: The following is a simplified derivation showing that the probability density function (pdf) for the normalized sample variance, $X = (n-1)S^2/\sigma^2$, is the χ^2-distribution with $\nu = n - 1$ degrees of freedom where n is the number of independent, normally distributed samples, σ^2 is the variance of each sample, and sample variance s^2 is defined in the standard way:

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}\left(X_i - \overline{X}\right)^2 \tag{1}$$

where the X_i are the samples, and $\overline{X}$ is the sample mean defined in the standard way:

$$\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i\,. \tag{2}$$

To improve clarity and focus attention on key ideas in the derivation, we assume the samples are drawn from a standard normal distribution with mean $\mu = 0$ and variance $\sigma^2 = 1$:

$$f_{X_i}(x_i) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x_i^2}\,. \tag{3}$$

Based on rules for linear combinations of random variables, the sample mean is normally distributed with variance $\sigma^2/n = 1/n$ since we are assuming $\sigma^2 = 1$.

$$f_{\overline{X}}(\overline{x}) = \frac{1}{\sqrt{2\pi/n}}e^{-\frac{1}{2}\frac{\overline{x}^2}{1/n}} = \frac{1}{\sqrt{2\pi/n}}e^{-\frac{1}{2}n\overline{x}^2} \tag{4}$$

The pdf for all the samples is an n-dimensional normal distribution [1].

$$f_{(x_1,\ldots,x_n)}(x_1,\ldots,x_n) = \frac{1}{(2\pi)^{n/2}}e^{-\frac{1}{2}\sum_{i=1}^{n}x_i^2} \tag{5}$$

With some manipulation of summations [2], we may show that the summation of the squared x_i's may be written in terms of the sample variance and sample mean:

$$\sum_{i=1}^{n} X_i^2 = (n-1)S^2 + n\overline{X}^2\,. \tag{6}$$

or

$$S^2 = \frac{1}{n-1}\left(\sum_{i=1}^{n} X_i^2 - n\overline{X}^2\right). \tag{7}$$

Using (6), we rewrite the n-dimensional normal distribution:

$$f_{(x_1,\ldots,x_n)}(x_1,\ldots,x_n) = \frac{1}{(2\pi)^{n/2}} e^{-\frac{1}{2}\left[(n-1)s^2 + n\overline{x}^2\right]}. \tag{8}$$

We find the pdf of $x = (n-1)s^2$ by taking the derivative of the cumulative distribution function.

$$\begin{aligned} f_X(x) &= f_{(n-1)S^2}\left((n-1)s^2\right) = \frac{d}{d(n-1)s^2} F\left((n-1)s^2\right) \\ &= \frac{d}{d(n-1)s^2} P\left((n-1)S^2 \le (n-1)s^2\right) = \frac{d}{d(n-1)s^2} P\left(S^2 \le s^2\right) \\ &= \frac{d}{d(n-1)s^2} P\left(S \le s\right) \end{aligned} \tag{9}$$

Given (8) and (9), our goal will be to express $P(S \le s)$ in terms of s, but our starting point is to find the cumulative probability by integrating the pdf of $(x_1, ..., x_n)$ over all the $(x_1, ..., x_n)$ that would give a sample variance that is less than or equal to s^2.

$$P\left(S \le s\right) = \iiint\limits_{\frac{1}{n-1}\left(\sum_{i=1}^{n} x_i^2 - n\overline{x}^2\right) \le s^2} f_{(x_1,\ldots,x_n)}(x_1,\ldots,x_n) dx_1 \ldots dx_n \tag{10}$$

or

$$P(S \le s) = \iiint\limits_{\sum_{i=1}^{n} x_i^2 \le (n-1)s^2 + n\overline{x}^2} \frac{1}{(2\pi)^{n/2}} e^{-\frac{1}{2}\sum_{i=1}^{n} x_i^2} dx_1 \ldots dx_n \tag{11}$$

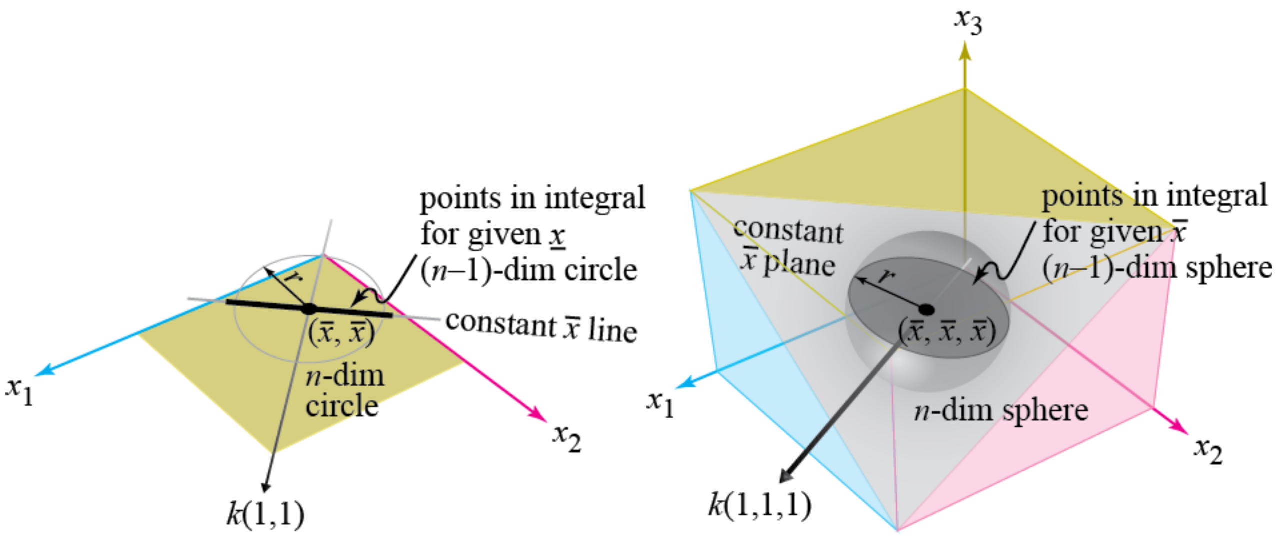
We observe that the pdf $f_{(x_1,\ldots,x_n)}(x_1,\ldots,x_n)$ is spherically symmetric, which suggests that we might be able to use spherical coordinates for our integral. However, the spherical symmetry of $f_{(x_1,\ldots,x_n)}(x_1,\ldots,x_n)$ is with respect to the origin, whereas we want to integrate over the $(x_1, ..., x_n)$ that are within a certain squared distance from $(\bar{x},\ldots,\bar{x})$. That is, $(n-1)s^2$ may be thought of as a measure of the squared distance from $(x_1, ..., x_n)$ to $(\bar{x}, ..., \bar{x})$:

$$\sum_{i=1}^{n}\left(X_i-\bar{X}\right)^2 \le (n-1)S^2 . \tag{12}$$

It follows that the $(x_1, ..., x_n)$ are points in an n-dimensional sphere centered at $(\bar{x},\ldots,\bar{x})$ at a squared distance of at most $(n-1)s^2$ or a radius of $r=\sqrt{n-1}\cdot s$.

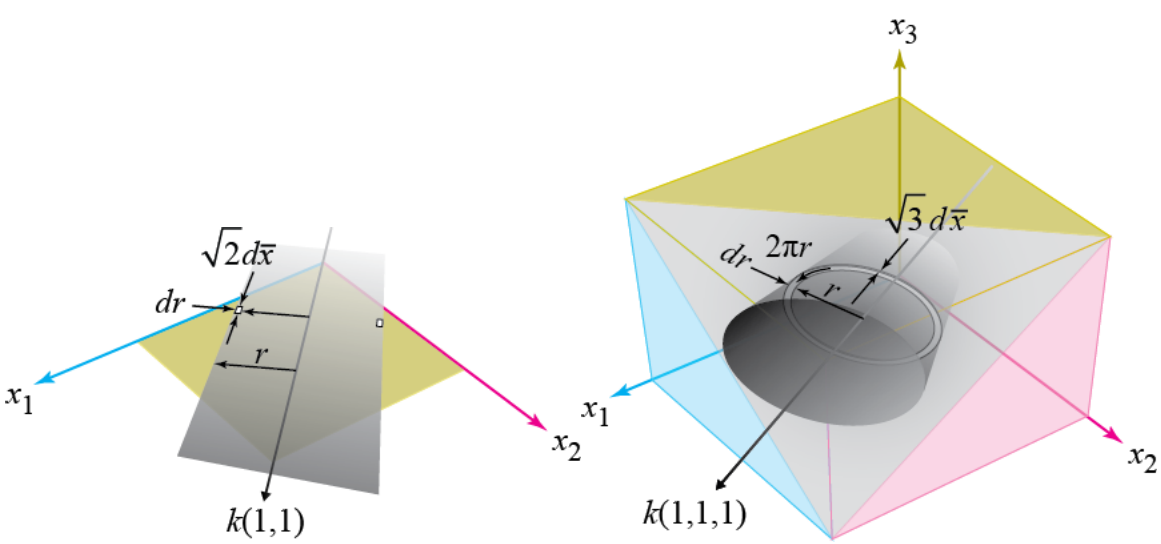
For a given $\bar{x}$, however, these $(x_1, ..., x_n)$ must also lie on the hyper-plane of points such that $\frac{1}{n}(x_1+\ldots+x_n)=\bar{x}$ since the average of the x_i is $\bar{x}$. This plane is perpendicular to $(\bar{x},\ldots,\bar{x})$ or a vector in the (1,1,1) direction. Thus, for a given $\bar{x}$, we are integrating over the intersection of an n-dimensional sphere of radius $r=\sqrt{n-1}\cdot s$ and a hyper-plane in n dimensions that is perpendicular to the (1,1,1) direction. The resulting intersection is an $(n-1)$-dimensional sphere. As shown in Fig. 1(a), for the case of $n = 2$, (2-dimensional space for X_1, X_2), the $(n-1)$-sphere is a 1-dimensional line of points on the constant $\bar{x}$ line, and as shown in Fig. 1(b), for the case of $n = 3$, (3-dimensional space for X_1, X_2, X_3), the $(n-1)$-sphere is a 2-dimensional circle of points on the constant $\bar{x}$ plane.

We may use $\bar{x}$ and r as orthogonal variables of integration. As we vary $\bar{x}$, the line of constant $\bar{x}$ moves a distance $\sqrt{n}\cdot\bar{x}$ in the (1,1) direction, and sphere of integrated points moves with it. This gives an extruded $(n-1)$-dimensional sphere as the region of integration. As shown in Fig. 2(a) for the case of $n = 2$, the region of integration is an infinite band in the (1,1) direction, and as shown in Fig. 2(b) for the case of $n = 3$, the region of integration is an infinite cylinder in the (1,1,1) direction.



(a) (b)

Fig. 1. Points to integrate in the r direction for calculation of $P(S \le s)$ at a given value of $\overline{x}$:
(a) 2-dimensional case, (b) 3-dimensional case.



(a) (b)

Fig. 2. Region of integration for calculation of $P(S \le s)$ in coordinates of $\overline{x}$ and r:
(a) 2-dimensional case is infinite band parallel to (1,1) direction,
(b) 3-dimensional case is infinite cylinder parallel to (1,1,1) direction.

For $n \geq 2$ dimensions, the above picture generalizes to the following change of variables:

$$dx_1 \ldots dx_n = \sqrt{n}\, d\overline{x} \cdot A_{n-1}(r) dr \,. \tag{13}$$

where $\overline{x}$ varies from $-\infty$ to ∞ and $A_{n-1}(r)$ is the surface area of an $(n-1)$-dimensional sphere of radius $r = \sqrt{n-1} \cdot s$.

From [3] we have the following formulas for sphere volumes and surface areas:

$$V_n(r) = \frac{S_n r^n}{n} \text{ is the volume of an } n\text{-dimensional sphere of radius } r \tag{14}$$

$$S_n(r) = \frac{2\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right)} \text{ is the surface area of an } n\text{-dimensional sphere of radius} = 1. \tag{15}$$

It follows that the surface area of an n-dimensional unit sphere is:

$$A_n(r) = S_n r^{n-1} = \frac{2\pi^{n/2} r^{n-1}}{\Gamma\left(\frac{n}{2}\right)} \,. \tag{16}$$

The gamma function has the following properties [4]:

$\Gamma(n) = (n-1)!$ for $n > 0$ a positive integer

$z\Gamma(z) = \Gamma(z+1)$ for all complex z except integers ≤ 0

$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

Using (16), we have:

$$A_{n-1}(r) = S_{n-1} r^{n-1} = \frac{2\pi^{(n-1)/2} r^{n-2}}{\Gamma\left(\frac{n-1}{2}\right)} \,. \tag{17}$$

We now have the following integral for $P(S \le s)$:

$$P(S \le s) = \iiint\limits_{\sum_{i=1}^{n}(x_i - \bar{x})^2 \le (n-1)s^2} \frac{1}{(2\pi)^{n/2}} e^{-\frac{1}{2}\sum_{i=1}^{n} x_i^2} dx_1 \ldots dx_n$$
$$= \int_{s=0}^{r=\sqrt{n-1}\cdot s} \int_{\bar{x}=-\infty}^{\bar{x}=\infty} \frac{1}{(2\pi)^{n/2}} e^{-\frac{1}{2}\left[r^2 + n\bar{x}^2\right]} \sqrt{n}\, d\bar{x} \frac{2\pi^{(n-1)/2} r^{n-2}}{\Gamma\left(\frac{n-1}{2}\right)} dr . \tag{18}$$

We separate variables, and perform the inner integration first (after ensuring that the inner integration is of a normal density function, thus yielding a value of unity).

$$P(S \le s) =$$
$$\int_{r=0}^{r=\sqrt{n-1}\cdot s} \left[\int_{\bar{x}=-\infty}^{\bar{x}=\infty} \frac{1}{\sqrt{2\pi / n}} e^{-\frac{1}{2} n\bar{x}^2} d\bar{x} \right] \frac{1}{(2\pi)^{(n-1)/2}} e^{-\frac{1}{2}r^2} \frac{2\pi^{(n-1)/2} r^{n-2}}{\Gamma\left(\frac{n-1}{2}\right)} dr \tag{19}$$

The value inside the square brackets is our integral (of a normal density function) that has a value of unity. Thus, we have

$$P(S \le s) = \int_{r=0}^{r=\sqrt{n-1}\cdot s} \frac{1}{(2\pi)^{(n-1)/2}} e^{-\frac{1}{2}r^2} \frac{2\pi^{(n-1)/2} r^{n-2}}{\Gamma\left(\frac{n-1}{2}\right)} dr . \tag{20}$$

We now use $\nu = n - 1$ as the "degrees of freedom" to simplify the expression and reflect the idea that the pdf is analogous to one for $n - 1$ variables.

$$P(S \le s) = \int_{r=0}^{r=\sqrt{n-1}/s} \frac{1}{(2\pi)^{\nu/2}} e^{-\frac{1}{2}r^2} \frac{2\pi^{\frac{\nu}{2}} r^{\nu-1}}{\Gamma\left(\frac{\nu}{2}\right)} dr \tag{21}$$

Fortunately, we will take the derivative of the cumulative distribution, so computing the integral is unnecessary. However, we do have to deal with a change of variables for the derivative.

As a preliminary to using the chain rule, we have the following calculations:

$$x = (n-1)s^2 = r^2 \tag{22}$$

so

$$r = \sqrt{x} \tag{23}$$

and

$$\frac{dr}{dx} = \frac{1}{2\sqrt{x}}. \tag{24}$$

Using the chain rule, we have the following result:

$$\begin{aligned} f_X(x) &= \frac{d}{dx}P(S \le s) = \frac{dr}{dx}\frac{d}{dr}P(S \le s \text{ i.e., } r = \sqrt{x}) \\ &= \frac{1}{2\sqrt{x}}\frac{d}{dr}P\left(S \le s; r = \sqrt{x}\right). \end{aligned} \tag{25}$$

The final derivative is the derivative of an integral, so the final derivative is just the integrand from (21):

$$f_X(x) = \begin{cases} \dfrac{1}{2\sqrt{x}}\dfrac{1}{(2\pi)^{\nu/2}}e^{-\frac{1}{2}r^2}\dfrac{2\pi^{\frac{\nu}{2}}r^{\nu-1}}{\Gamma\left(\frac{\nu}{2}\right)} & x > 0 \\ 0 & \text{otherwise} \end{cases} \tag{26}$$

or, since $r^2 = x$ and several constants cancel out,

$$f_X(x) = \begin{cases} \dfrac{1}{2^{\nu/2}\Gamma\left(\frac{\nu}{2}\right)}e^{-\frac{1}{2}x}x^{\frac{\nu}{2}-1} & x > 0 \\ 0 & \text{otherwise} \end{cases}. \tag{27}$$

In conclusion, the distribution of $x = (n - 1)s^2$ when $\sigma^2 = 1$ is a chi-squared distribution. Without proof, we state the following result when $\sigma^2 \neq 1$:

The probability density function of $x = \frac{(n-1)S^2}{\sigma^2}$ is a chi-squared distribution with $\nu = n - 1$ degrees of freedom [2]:

$$f_{\chi^2,\nu}(x) = \begin{cases} \frac{1}{2^{\frac{\nu}{2}}\Gamma\left(\frac{\nu}{2}\right)} x^{\frac{\nu}{2}-1} e^{-\frac{x}{2}} & x > 0 \\ 0 & \text{otherwise} \end{cases} \tag{28}$$

REF: [1] "The Multivariate Normal Distribution." http://www.math.uah.edu/stat/special/MultiNormal.html

[2] Ronald E. Walpole, Raymond H. Myers, Sharon L. Myers, and Keying Ye, *Probability and Statistics for Engineers and Scientists,* 8th Ed., Upper Saddle River, NJ: Prentice Hall, 2007.

[3] Weisstein, Eric W. "Hypersphere." From MathWorld--A Wolfram Web Resource. http://mathworld.wolfram.com/Hypersphere.html

[4] Weisstein, Eric W. "Gamma Function." From MathWorld--A Wolfram Web Resource. http://mathworld.wolfram.com/GammaFunction.html