EX: A sample-and-hold circuit is used in an A/D converter to store a voltage on a capacitor while it is being translated into a binary number. As with any capacitor, the stored charge on the capacitor leaks away over time. The loss of voltage is modeled by a capacitor discharge equation:

$$V = v_0 e^{-T/RC}$$

where

$V \equiv$ voltage on capacitor when A/D conversion is complete (volts)

$v_0 \equiv$ initial voltage on capacitor = 1 V for this problem

$T \equiv$ time required for A/D conversion = gaussian distributed random variable with
mean 20 ns and variance $(2 \text{ ns})^2$

$RC \equiv$ time constant for leakage = 6 μs

a) Find the probability density function, $f_V(v)$, of the voltage on the capacitor at the end of the A/D conversion.

b) Find the probability that the voltage on the capacitor droops enough for a 1-bit error in an 8-bit value. In other words, find $P(V \le v_0 \cdot 255/256)$. Hint: translate the problem into that of finding a probability for a gaussian random variable and use Table A.3 in the course text to find that probability.

SOL'N: a) Because T is gaussian (or normal) and appears in the exponent, the form of V is almost a lognormal distribution. The form of the lognormal probability density function (pdf), [1], requires that the entire exponent be gaussian:

$$X = e^Y$$

where Y is gaussian distributed has lognormal pdf, $f_X(x)$, as follows

$$f_X(x) = \begin{cases} \dfrac{1}{x\sqrt{2\pi\sigma_Y^2}} e^{-[\ln(x)-\mu_Y]^2/2\sigma_Y^2} & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

where

$$\mu_X = e^{\mu_Y + \frac{\sigma_Y^2}{2}} \qquad \sigma_X^2 = e^{2\mu_Y + \sigma_Y^2}\left(e^{\sigma_Y^2} - 1\right)$$

In the present problem, we have $Y = \frac{-T}{RC}$ and V replaces X. This is a linear transformation of a gaussian distribution, which is again a gaussian distribution. The mean and variance of this gaussian are as follows:

$$\mu_Y = \frac{-1}{RC}\mu_T = \frac{-1}{6\mu\text{s}} \cdot 20\text{ns} = -\frac{10}{3}\text{m}$$

and

$$\sigma_Y^2 = \left(\frac{-1}{RC}\right)^2 \sigma_T^2 = \left(\frac{-1}{6\mu\text{s}}\right)^2 (2\text{ns})^2 = \left(\frac{1}{3}\text{m}\right)^2$$

Replacing X with V in the lognormal pdf, we have our final expression for $f_V(v)$.

$$f_V(v) = \begin{cases} \frac{1}{v\sqrt{2\pi\sigma_Y^2}} e^{-[\ln(v) - \mu_Y]^2/2\sigma_Y^2} & v > 0 \\ 0 & \text{otherwise} \end{cases}$$

b) We find $P(V \le v_0 \cdot 255/256)$ by substituting for V in terms of T and using the cumulative distribution for T:

$$P(V \le v_0 \cdot 255/256) = P\left(e^{\frac{-T}{RC}} \le v_0 \cdot 255/256\right)$$

or

$$P(V \le v_0 \cdot 255/256) = P\left(T \ge -RC \cdot \ln(v_0 \cdot 255/256)\right)$$

or

$$P(V \le v_0 \cdot 255/256) = 1 - F_T\left(t = -RC \cdot \ln(v_0 \cdot 255/256)\right)$$

or

$$P(V \le v_0 \cdot 255/256) = 1 - F_T\left(t = 6\mu\text{s} \cdot 3.914\text{m}\right) = 1 - F_T\left(t = 23.48\text{n}\right)$$

Now we convert T to is equivalent value for a standard gaussian (or normal) distribution:

$$Z = \frac{T - \mu_T}{\sigma_T}$$

This means we use the value of t to find the value of z in the cumulative distribution, $F_Z(z)$:

$$P(V \leq v_0 \cdot 255/256) = 1 - F_Z\left(z = \frac{23.48\text{n} - 20\text{n}}{2\text{n}} = 1.740\right)$$

Using a table for the cumulative distribution of the standard gaussian, [1], we lookup the value of the probability:

$$F_Z(1.740) = 0.9591$$

Thus, we have the following final result:

$$P(V \leq v_0 \cdot 255/256) = 1 - 0.9591 = 0.0409$$

REF: [1] Ronald E. Walpole, Raymond H. Myers, Sharon L. Myers, and Keying Ye, *Probability and Statistics for Engineers and Scientists,* 8th Ed., Upper Saddle River, NJ: Prentice Hall, 2007.