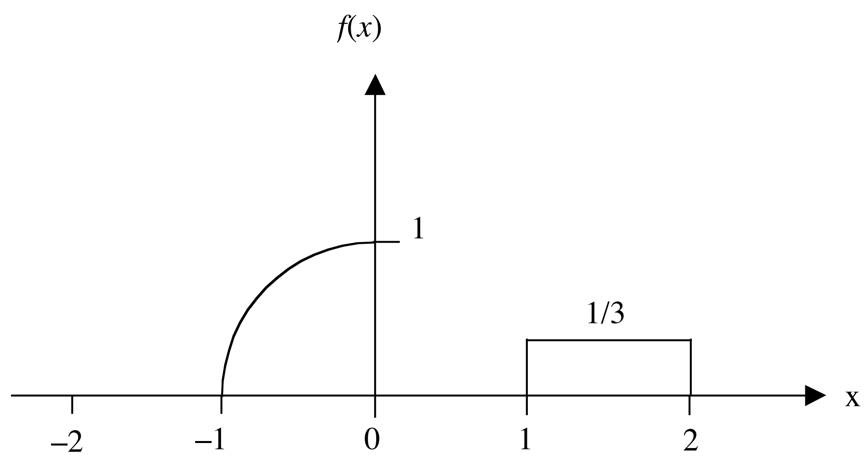
EX:



A probability density function is shown above and is described by the following equation:

$$f(x) = \begin{cases} 1 - x^2 & -1 \le x \le 0 \\ 1/3 & 1 \le x \le 2 \\ 0 & \text{otherwise} \end{cases}$$

a) Plot the cumulative distribution function, $F(x)$, for X.

b) Find $P(1/2 \le x \le 3/2)$

c) Calculate σ^2 for X.

SOL'N: a) $F(x)$ is the integral from $-\infty$ to x of $f(x)$ or, equivalently, the area under $f(x)$ to the left of x:

$$F(x) = \int_{-\infty}^{x} f(x)dx$$

Because the definition of $f(x)$ changes with x, we break the integral into segments and use only the segments that are left of x:

$$F(x)=\begin{cases} 0 & x\le -1 \\ \int_{-1}^{x} 1-x^2 dx & -1\le x\le 0 \\ \int_{-1}^{0} 1-x^2 dx & 0\le x\le 1 \\ \int_{-1}^{0} 1-x^2 dx+\int_{0}^{x} 1/3\, dx & 1\le x\le 2 \\ 1 & x\ge 2 \end{cases}$$

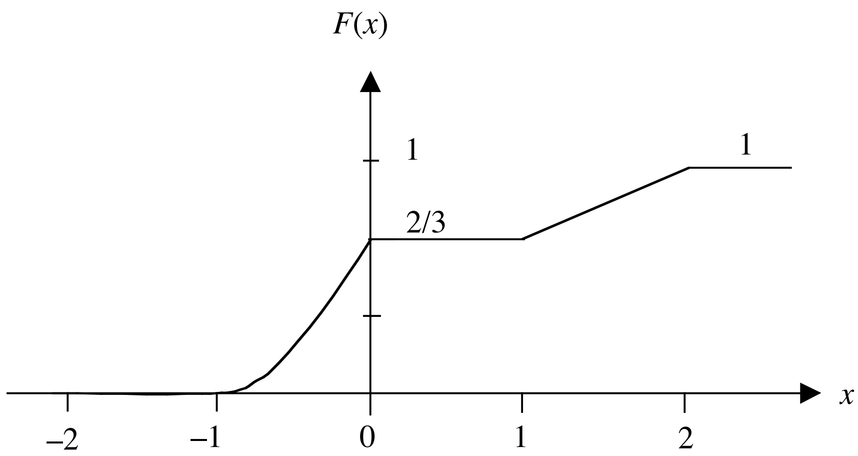
Note that the last entry is 1 because we know the total area under $f(x)$ equals one. Calculating the integrals, we have

$$F(x)=\begin{cases} 0 & x\le -1 \\ x-\left.\dfrac{x^3}{3}\right|_{-1}^{x} & -1\le x\le 0 \\ x-\left.\dfrac{x^3}{3}\right|_{-1}^{0} & 0\le x\le 1 \\ x-\left.\dfrac{x^3}{3}\right|_{-1}^{0}+\left.\dfrac{1}{3}x\right|_{0}^{x} & 1\le x\le 2 \\ 1 & x\ge 2 \end{cases}$$

or

$$F(x)=\begin{cases} 0 & x\le -1 \\ x+1-\left(\dfrac{x^3}{3}+\dfrac{1}{3}\right)=\dfrac{2}{3}+x-\dfrac{x^3}{3} & -1\le x\le 0 \\ \dfrac{2}{3} & 0\le x\le 1 \\ \dfrac{2}{3}+\dfrac{1}{3}x & 1\le x\le 2 \\ 1 & x\ge 2 \end{cases}$$

Unless $F(x)$ contains delta functions, the plot of $F(x)$ must be continuous.



b) By definition, $F(x) = P(X \le x)$. It follows that

$$P(1/2 \le x \le 3/2) = F(3/2) - F(1/2)$$

From part (a), we have $F(3/2) = 5/6$ and $F(1/2) = 2/3$. Thus,

$$P(1/2 \le x \le 3/2) = 5/6 - 2/3 = 1/6.$$

Another way to obtain this result is to integrate the probability density function:

$$P(1/2 \le x \le 3/2) = \int_{1/2}^{3/2} f(x)dx = \int_{1}^{3/2} \frac{1}{3}dx = \frac{1}{3}x\Big|_{1}^{3/2} = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$

c) The variance, σ^2, is given by the formula $\sigma^2 = E(X^2) - \mu^2$. First, we calculate μ:

$$\mu = \int_{-\infty}^{\infty} xf(x)dx = \int_{-1}^{0} x(1-x^2)dx + \int_{1}^{2} x\frac{1}{3}dx$$

or

$$\mu = \left(\frac{x^2}{2} - \frac{x^4}{4}\right)\Bigg|_{-1}^{0} + \frac{1}{3}\frac{x^2}{2}\Bigg|_{1}^{2} = -\frac{1}{2} + \frac{1}{4} + \frac{4}{6} - \frac{1}{6} = \frac{1}{4}$$

Second, we calculate $E(X^2)$:

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)dx = \int_{-1}^{0} x^2(1-x^2)dx + \int_{1}^{2} x^2 \frac{1}{3} dx$$

or

$$E(X^2) = \left(\frac{x^3}{3} - \frac{x^5}{5}\right)\Bigg|_{-1}^{0} + \frac{1}{3}\frac{x^3}{3}\Bigg|_{1}^{2} = \frac{1}{3} - \frac{1}{5} + \frac{8}{9} - \frac{1}{9} = \frac{15-9+40-5}{45} = \frac{41}{45}$$

Combining results, we have

$$\sigma^2 = \frac{41}{45} - \left(\frac{1}{4}\right)^2 = \frac{656-45}{720} = \frac{611}{720} \approx 0.849$$