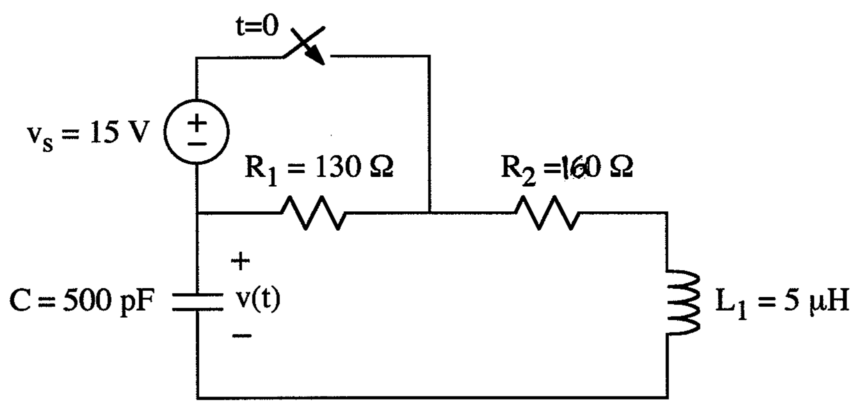
Ex:



After being open for a long time, the switch closes at t = 0.

Find v(t) for t > 0.

Sol'n: First, we find the characteristic roots for the circuit. After $t=0$ we have a series RLC circuit with a voltage source. R_1 is bypassed so we use $R=R_2$.

For series RLC, $\alpha = \dfrac{R}{2L}$, $\omega_0^2 = \dfrac{1}{LC}$

$$\alpha = \frac{160\,\Omega}{2 \cdot 5\,\mu H}, \quad \omega_0^2 = \frac{1}{5\,\mu H \cdot 500\,pF}$$

$$\alpha = 16\text{ M r/s}, \quad \omega_0^2 = 400\text{ M}^2(\text{r/s})^2$$

$$\omega_0 = 20\text{M r/s}$$

$\omega_0 > \alpha$ so we have underdamped case:

$$\omega_d = \sqrt{\omega_0^2 - \alpha^2} = \sqrt{(20\text{M})^2 - (16\text{M})^2}\text{ r/s}$$

$$\omega_d = 12 \text{ M r/s}$$

Note: $\omega_d > \alpha$ can happen but $\omega_d < \omega_o$ always.

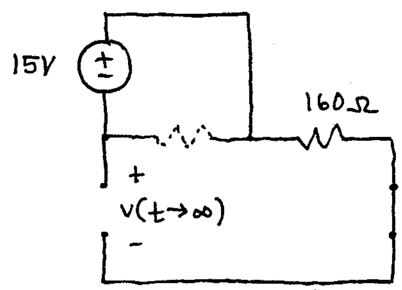
Our sol'n form is

$$v(t) = A_1 e^{-\alpha t} \cos \omega_d t + A_2 e^{-\alpha t} \sin \omega_d t + A_3 .$$

Second, we find $A_3 = v(t \to \infty)$.

As $t \to \infty$, C = open L = wire switch closed

model:
$(t \to \infty)$



No current in R so no V-drop for R. Thus, 15V from source is across C.

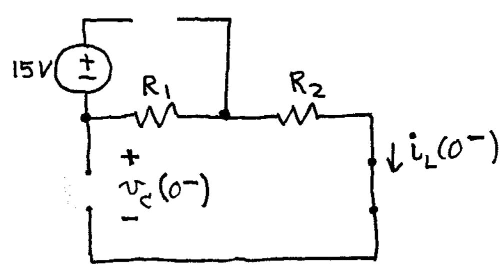
$$v(t \to \infty) = -15V$$

$$\therefore \; A_3 = -15V$$

Third, we find $i_L(t=0^-)$ and $v_c(t=0^-)$ as a precurser to finding $v(0^+)$ and $\left.\frac{dv}{dt}\right|_{t=0^+}$.

At $t=0^-$, C = open L = wire switch open

model:
$(t=0^-)$

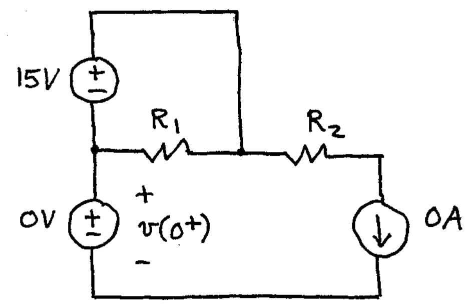


Source not connected so $i_L(0^-)=0$ and $v_c(0^-)=0$.

Fourth, we have $i_L(0^+) = i_L(0^-)$ and $v_C(0^+) = v_C(0^-)$ since these are energy variables ($w = \frac{1}{2} L i^2$ and $w = \frac{1}{2} C v^2$) that cannot change instantly.

At $t = 0^+$, we model L as a current source with value $i_L(0^+)$ and C as a voltage source with value $v_C(0^+)$.

model:
($t = 0^+$)



Fifth, we solve the circuit to find $v(0^+)$. Here, we have that $v(0^+) = 0V$ without doing any additional work.

For the form of sol'n we are using, we have

$$v(0^+) = A_1 e^{-\alpha t} \cos(\omega_d t) + A_2 e^{-\alpha t} \sin(\omega_d t) + A_3 \Big|_{t=0^+}$$

$$'' \quad = A_1 \cdot e^{-0^+} \cos(0) + A_2 e^{-0^+} \sin(0) + A_3$$

$$'' \quad = A_1 \cdot 1 \cdot 1 + A_2 \cdot 1 \cdot 0 + A_3$$

$$'' \quad = A_1 + A_3$$

Equating the known value of $v(0^+) = 0V$ with the symbolic sol'n, we conclude that:

$$0V = A_1 + A_3 = A_1 - 15V \quad \text{or} \quad A_1 = 15V$$

Sixth, we use the circuit model at $t=0^+$ to find $\left.\frac{d}{dt} v(t)\right|_{t=0^+}$.

The method we use to find any derivative value at $t=0^+$ is to write an expression for $v(t)$ in terms of only $i_L(t)$ and $v_C(t)$ plus component values.

Here, we have the simple result that

$$v(t) = v_C(t)$$

Now we differentiate this entire eq'n:

$$\frac{dv(t)}{dt} = \frac{dv_C(t)}{dt}$$

From $i_C(t) = C\frac{dv_C(t)}{dt}$ we have $\frac{dv_C(t)}{dt} = \frac{i_C(t)}{C}$.

(Although we don't require it here, we also have $v_L(t) = L\frac{di_L(t)}{dt}$ or $\frac{di_L(t)}{dt} = \frac{v_L(t)}{L}$.)

Thus $\left.\frac{dv(t)}{dt}\right|_{t=0^+} = \left.\frac{dv_C(t)}{dt}\right|_{t=0^+} = \left.\frac{i_C(t)}{C}\right|_{t=0^+}$.

We use our model for $t=0^+$ to find $i_C(t=0^+)$. (See above.) From the model, $i_C(0^+) = 0A$ since C is in series with a $0A$ source.

$$\therefore \left.\frac{dv(t)}{dt}\right|_{t=0^+} = \left.\frac{i_C(t)}{C}\right|_{t=0^+} = \frac{0A}{C} = 0 \ V/s$$

Equating this known value of $\left.\frac{dv(t)}{dt}\right|_{t=0^+}$ with $\frac{d}{dt}$ of the symbolic sol'n,

we have

$$0V = \left.\frac{d}{dt} v(t)\right|_{t=0^+} = \frac{d}{dt}\left[A_1 e^{-\alpha t}\cos(\omega_d t) + A_2 e^{-\alpha t}\sin(\omega_d t) + A_3\right]\Bigg|_{t=0^+}$$

$$'' = A_1(-\alpha)e^{-\alpha t}\cos(\omega_d t) + A_1 e^{-\alpha t}\omega_d[-\sin(\omega_d t)] + A_2(-\alpha)e^{-\alpha t}\sin(\omega_d t) + A_2 e^{-\alpha t}\omega_d\cos(\omega_d t)\Big|_{t=0^+}$$

$$'' = A_1(-\alpha)\,1\cdot 1 + A_1\cdot 1\cdot\omega_d\cdot 0$$
$$A_2(-\alpha)\,1\cdot 0 + A_2\cdot 1\cdot\omega_d\cdot 1$$

$$0V = A_1(-\alpha) + A_2\omega_d$$

Thus, $A_2 = \frac{A_1\alpha}{\omega_d} = \frac{15V\cdot 16\,Mr/s}{12\,Mr/s} = 20V$

$$\therefore v(t>0) = 15V e^{-16Mt}\cos(12Mt) + 20V e^{-16Mt}\sin(12Mt) - 15V$$

Check: $v(0^+) = 15V - 15V = 0V$ ✓

When $\left.\frac{dv}{dt}\right|_{t=0^+} = 0$ we should have $A_1\alpha = A_2\omega_d$ ✓