May 1990 Real Analysis — Convergence — Almost everywhere
Pointwise

Neil E Cotter

$\langle f_n \rangle \equiv$ sequence of functions on domain D

$\langle f_n \rangle$ converges to f <u>almost everywhere</u> (a.e.) $\equiv$

There is a set M of measure zero such that $\langle f_n \rangle$ converges to f pointwise on $D \sim M$ (where $D \sim M$ means set D subtract set M).

ex: $f_n(x) = \begin{cases} [\cos 2\pi x][1-\frac{1}{n}] & x \text{ irrational} \\ 3 & x \text{ rational} \end{cases}$ $\quad x \in D = [0,1]$

Then $\langle f_n \rangle \to f = \cos 2\pi x$ a.e.

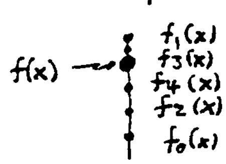
Here set M is the rational #'s in $[0,1]$. At rational #'s, $f_n(x) = 3 \neq \cos 2\pi x$. At every other x in D, however, we get $f_n(x) = (1-\frac{1}{n}) \cos 2\pi x$ which converges to f as $n \to \infty$.

$\langle f_n \rangle$ converges to f <u>pointwise</u> $\equiv$

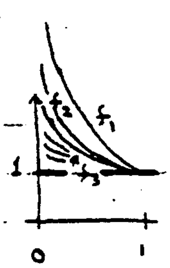
For every x in D, $f(x) = \lim\limits_{n\to\infty} f_n(x)$,

i.e. given $x \in D$, then if $\epsilon > 0$ there is an integer N such that $|f(x) - f_n(x)| < \epsilon$ for all $n \geq N$. Note that N changes with the choice of ϵ <u>and</u> x.

For a particular x, the values of $f_n(x)$ form a sequence of points converging to the point $f(x)$:



For $n \geq N$, i.e. at some point in the sequence, $f_n(x)$ is within ϵ of $f(x)$ for all f_n.



Note: the rate of convergence may be different at each x. Consider $f_n(x) = \frac{1}{x^{1/n}}$ on $D = (0, 1]$. Converges slowly for x near zero. Converges to $f(x) = 1$.