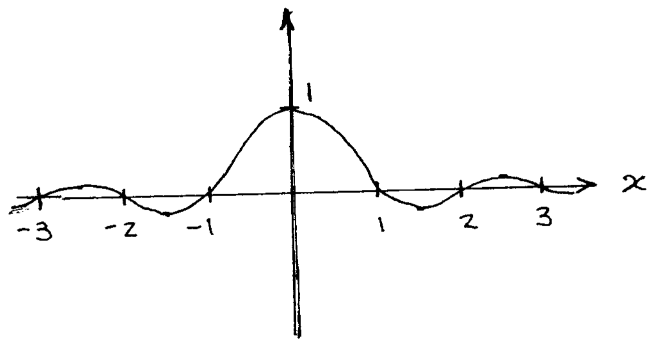
Sinc Functions

def: $\text{sinc}(x) \equiv \dfrac{\sin(\pi x)}{\pi x}$



not'n: $\text{sinc}_{k,j}(t) \equiv 2^{j/2}\,\text{sinc}(2^j t - k) \equiv S_{k,j}(t)$

thm: The set of functions $\{S_{k,j}(t) : k = -\infty, \ldots, \infty\}$ (for one integer value of j) is

a complete orthonormal basis for $W(\pi/h) \equiv$ finite-energy analytic functions, i.e. for functions $f(t)$ that are differentiable at every point in the complex plane, (analytic), we have

$$\int_{-\infty}^{\infty} |f(t)|^2\, dt < \infty$$

and

$$|f(z)| \leq C e^{\pi |z|/h} \quad \text{(for some } C > 0)$$

for all $z \in \mathbb{C}$ (complex plane), [1].

Note: The condition of being analytic implies the existence of derivatives of every order, $d^n f(t)/dt^n$, which is a stronger condition than being a real-valued differentiable function.

note: $\varphi(t) = \text{sinc}(t)$ is a suitable scaling function for the sinc function being used as a wavelet:

$$\Psi(t) = 2\varphi(2t) - \varphi(t)$$

or $$\Psi(t) = 2\,\frac{\sin 2\pi t}{2\pi t} - \frac{\sin \pi t}{\pi t}$$

This wavelet does not have compact support in the time domain, but it has finite support in the frequency domain.

$$\Phi(\omega) = \int_{-\infty}^{\infty} \varphi(t)\, e^{-j\omega t}\, dt = \begin{cases} 1 & -\pi < \omega < \pi \\ 0 & \text{otherwise} \end{cases}$$

$$\Psi(\omega) = \int_{-\infty}^{\infty} \Psi(t)\, e^{-j\omega t}\, dt = \begin{cases} 1 & \pi < |\omega| < 2\pi \\ 0 & \text{otherwise} \end{cases}$$

note: Coefficients for $\varphi(t)$ are $h(n) = \text{sinc}\left(\frac{n}{2}\right)$

tool: At any one chosen level of resolution j, since $\{ S_{k,j}(t) : k = -\infty, \dots, \infty \}$ is a complete orthonormal basis, we have

$$f(t) = \sum_{k=-\infty}^{\infty} a_k\, S_{k,j}(t)$$

where $a_k = \langle f(t), S_{k,j}(t) \rangle = f(k/2^j)$

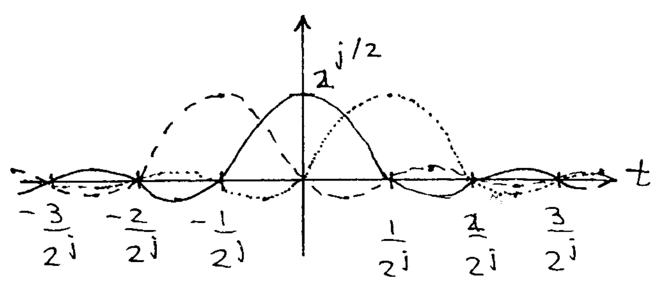
or $a_k = \int_{-\infty}^{\infty} f(t)\, S_{k,j}(t)\, dt = f(k/2^j)$

comment: The sinc functions in the set of $S_{k,j}$ for a given j form

a basis for functions that are bandlimited. For exact approximations, the value of j must be large enough to give a bandwidth larger than the bandwidth of $f(t)$.

We also conclude that any bandwidth limited signal may be represented as a summation of sinc functions positioned at regularly-spaced intervals.

note: The coefficients in a sinc-function expansion are values of $f(t)$ sampled at times $t = k/2^j$. This follows from the fact that the sinc functions are equal to zero at adjacent samples:



When these sinc functions are multiplied by a_k coefficients and summed, the only contribution to the sum for the point $t = k/2^j$ comes from the sinc function centered at $k/2^j$.

It follows that the value of a_k must be $f(t = k/2^j)$, which is the value of $f(t)$ at the k^{th} sample point.

note: Since the sample values of $f(t)$ are the a_k's in the expansion

$$f(t) = a_k \, S_{k,j}(t)$$

it follows that we can obtain all values of $f(t)$ between sample points from the values of $f(t)$ at the sample points. This is another example of the Nyquist criterion, which states that a function that is bandlimited may be reconstructed from samples spaced closer together than

$$\Delta t = 1/2 \cdot \text{bandwidth}$$